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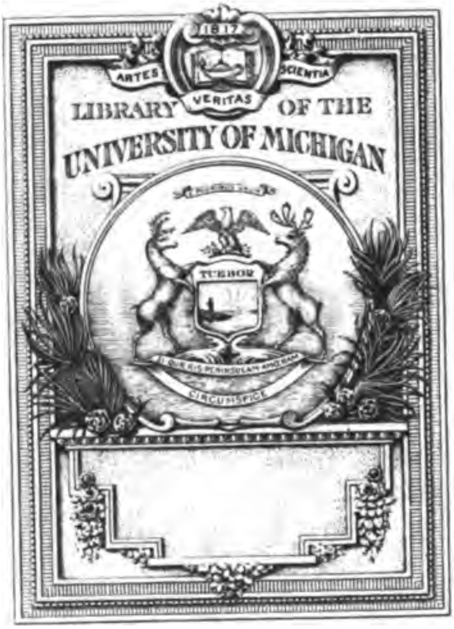

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ON

CONIC SECTIONS.

A NEW EDITION.

ADVERTISEMENT.

THE knowledge of the CONIC SECTIONS is ſo conſiderable a part of Geometry, and of ſuch extenſive uſefulneſs thro' almoſt all the practical as well as ſpeculative branches of that art, by which we conſider magnitude, motion or extenſion, that it would be meer vanity to pretend to a proficiency in the more abſtruſe parts of thoſe ſtudies, or what we may call high mathematics, without a thorough acquaintance with the properties of the three ſections of a cone, which are the ſubject of this treatiſe. The ſpeculations they afford are indeed ſurpriſing as well as entertaining, and require no ſmall degree of attention and capacity. Hence the antients, who ſeem leſs laviſh of their praiſes than the men of this generation, have honour'd Apollonius the Pergæan with the title of the great geometrician, for his exquiſite treatiſe on this ſubject, which indeed may be reckon'd the elements of a higher ſort of geometry, as the circle and triangle, which may be alſo conceived as ariſing from the ſection of a cone, are conſidered by Euclid in his laying the foundations of the loweſt and moſt ſimple kind of geometry, whoſe conſtruction extends not beyond the rule and compaſs. It would be eaſy to enlarge on the uſefulneſs and neceſſity of this branch of the mathematics, in order to any conſiderable progreſs in thoſe ſtudies; but we ſhall only add, that as this book has been recommended, by good judges, for one of the beſt, as well as plaineſt, in its kind, we thought we could not do a more effectual ſervice to the young ſtudent in theſe ſciences, than to oblige him (the firſt impreſſion being become very ſcarce) with a new, and much cheaper edition, which is printed in quarto, to accommodate thoſe who ſhall think fit to bind it up with the Miſcellanea Curioſa Mathematica. As for the reſt the reader is referr'd to the author's Preface.

A TREATISE OF CONIC SECTIONS.

DEDICATED TO THE

PROVOST, FELLOWS, and SCHOLARS of
the College of *DUBLIN*.

BY *ROBERT STEELL*.

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PREFACE.

MY deſign in this treatiſe, being to demonſtrate the principal properties of the *Conic Sections*, in the moſt eaſy manner; I ſhall not confine myſelf ſtrictly either to the *analytic*, or *ſynthetic* method, but ſhall uſe both indifferently, as I ſhall find the one or the other beſt anſwer that end.

Neither ſhall I ſcruple to borrow, or alter what I find for my purpoſe, in the writings of others on the ſame ſubject.

Mr *Walton*, on peruſing the manuſcript, was pleas'd to communicate ſome properties, which the reader will find in the *Italic* character.

The

The *Signs* and *Characters*, uſed in this Treatiſe.

$\therefore$	Therefore.
$\parallel$	Parallel.
$\angle$	Angle.
∟	Right-angle.
▯	Parallelogram.
$\triangle$	Triangle.
$\overline{a+b}\vert^{2}$	Square of $a+b$.
$\backsim$	Difference, or exceſs.
$t^{\frac{3}{2}}$	Square root of the cube of t.
$a:b::c:d$	Proportionality.

Conic Sections.

PART I.

Of the PARABOLA.

The GENESIS.

*IF from a point V, in any indefinite right line, there be taken $VD = VK$, and, from the point K as a center, with the diſtance DG, you interſect CM, perpendicular to DG, in the points C and M, thoſe points will be in the curve of a Parabola. Fig. 1.

DEFINITIONS.

1. The point V is the vertex, and K the focus of the Parabola.
2. The right line DG, paſſing thro' the focus, is called the axis.
3. A right line perpendicular to the axis, and terminated by the curve, is an ordinate to the axis, as GC.
4. The diſtance (in the axis) from the vertex to the interſection of the ordinate, is called the abſciſſa of that ordinate, as VG.
5. A right line drawn from any point of the curve, and parallel to the axis, is called a diameter, as CY; and the point in the curve, from which it is drawn, is called the vertex of that diameter.

PROPOSITION I.

THE ſquare of any ordinate, is equal to the rectangle of the abſciſſa of that ordinate into quadruple the diſtance of the focus from the vertex, that is, $GCq = VG \times 4KV$. Fig. 1.

DEMONSTRATION.

Put $KV = VD = q$, $VG = x$, and $GC = y$; then, (by the *Geneſis*) $GK = q \backsim x$, and $DG = CK = q + x$; but (by 47 *E.* 1.) $KCq = KGq + GCq$, that is, $q^2 + 2qx + x^2 = q^2 - 2qx + x^2 + y^2$, or $4qx = y^2$, *i. e.* $GCq = VG \times 4KV$. *Q. E. D.*

COR-

COROLLARY I.

The ſquares of the ordinates are to each other as their abſciſſas, becauſe $Y^2=4qX$, and $y^2=4qx$, therefore $Y^2 : y^2 :: (4qX : 4qx ::) X : x$.

DEFINITION.

The quadruple of the focal diſtance is called the parameter of the axis, and is a third proportional to any abſciſſa and its ordinate. For by putting $4q=p$, it will be $px=y^2$, therefore $x : y :: y : p$.

COROLLARY II.

The ordinate, which paſſes through the focus, is equal to half the parameter of the axis. For, in this caſe, $x=q$, therefore (by the propoſition) $4q^2=y^2$, and $y=(2q=)\frac{1}{2}p$.

PROPOSITION. II.

Fig. 2. AS the parameter of the axis is to the ſum of any two ordinates, ſo is their difference to the difference of their abſciſſas; that is, $p : IN :: NO : NC$.

DEMONSTRATION.

Put $HO=Y$, $GC=y$, $VH=X$, and $VG=x$; then (by *Prop.* 1.) $pX=Y^2$, and $px=y^2$, therefore $pX-px=Y^2-y^2$, and $p : Y+y :: Y-y : X-x$; that is, $p : IN :: NO : NC$. *Q. E. D.*

PROPOSITION III.

Fig. 3. IF from the vertex a right line be drawn ſo as to cut the curve, and continued till it meet any ordinate produc'd, it will be, as the parameter of the axis, is to the ordinate drawn from the interſection with the curve, ſo is the produc'd ordinate to its abſciſſa, that is, $p : GM :: HS : HV$.

DEMONSTRATION.

Let $HS=b$, $VG=x$, $VH=X$, $HO=Y$, and $GM=y$. Then (by *Prop.* 1.) $y^2 : Y^2 :: (x : X ::$ by ſimilar △'s$)\ y : b$; therefore $\frac{byy}{y}=Y^2=pX$, or $by=pX$; that is, $p : y :: b : X$; or, $p : GM :: HS : HV$. *Q. E. D.*

PROPOSITION IV.

IF from any point D, in the axis produced, a right line be drawn interſecting the curve in two points C and I, and the ordinates C P, I H be drawn from the ſaid interſections; V D, will be a mean proportional between V P and V H.

DEMONSTRATION.

Put $VD=b$, $VP=x$, $VH=X$, then $PD=b+x$, and $HD=b+X$; but (from ſimilar Δ's) $PDq : HDq :: PCq : IHq ::$ (by *Prop.* 1.) $PV : HV$; that is, $b^2+2bx+x^2 : b^2+2bX+X^2 :: x : X$. Therefore $\overline{X-x}\times b^2=\overline{X-x}\times Xx$; or $b^2=Xx$, that is, $x : b :: b : X$, or $VP : VD :: VD : HV$. *Q. E. D.*

PROPOSITION V.

IF any right line touch the curve, and an ordinate be drawn from the point of contact, then, I ſay, the abſciſſa of that ordinate ſhall be equal to the diſtance (in the axis produced) from the vertex of the curve to the interſection of the tangent, that is, $GV=VT$. Fig. 4.

DEMONSTRATION.

Let rF be an indefinitely ſmall part of the curve, and continued to T, draw $rs \parallel$ to the ordinate, and $Fp \parallel$ to the axis, and let $Fp=n$, $rp=m$, $VT=a$; and the other ſymbols as uſual. Then $Vs=x+n$, and $rs=y+m$; and by ſimilar Δ's $m : n :: y : x+a$; therefore $n\times\frac{y}{m}=\left(\frac{ny}{m}=\right)x+a$; and (by *Prop.* 1.) $p\times Vs=\overline{sr}\rceil^2$; and $p\times VG=GFq$, that is, $pn+px=y^2+2ym+m^2$, and $px=y^2$, therefore $y^2+2ym-pn=(px=)\ y^2$; that is, $n=\frac{2ym}{p}$; and conſequently, $x+a=\left(\frac{2ym}{p}\times\frac{y}{m}=\frac{2y^2}{p}=\frac{2px}{p}=\right)2x$. And $a=x$, or, $VT=GV$. *Q. E. D.*

PROPOSITION VI.

IF, from the point of contact, a right line be drawn to the focus, it will be equal to the diſtance (in the axis produc'd) from the focus to the interſection of the tangent, that is, $KF=KT$. Fig. 4.

DEMON-

DEMONSTRATION.

By *Prop.* 5. $GT=2x$, and (by *Prop.* 1.) $KG=x-\frac{1}{4}p$, $\therefore KT=(GT-KG=2x-x+\frac{1}{4}p=x+\frac{1}{4}p=$(by the *Geneſis*) KF. *Q.E.D.*

PROPOSITION VII.

IF, to the tangent, from the point of contact; a perpendicular be drawn, and produced to meet the axis, then the diſtance in the axis from that point, to the ordinate drawn from the point of contact, that is, the *ſubnormal*, is equal to half the parameter of the axis, Fig. 5. that is, $QG=\frac{1}{2}p$.

DEMONSTRATION.

For QG put b, then (by 8 *Eu.* 6.) $GT : GC :: GC : GQ$; that is, $2x : y :: y : b$, $\therefore 2xb=(y^2=)px$, whence $b=\frac{1}{2}p$, or, $QG=\frac{1}{2}p$. *Q. E. D.*

PROPOSITION VIII.

THE diſtances from the focus to the point of contact, from the focus to the interſection of the tangent with the axis, and from the focus to the end of the ſubnormal are equal, that is, $FC=FT=FQ$.

DEMONSTRATION.

Fig. 5. By the *Geneſis* $GF=x-\frac{1}{4}p$, and (by *Prop.* 7th.) $QG=\frac{1}{2}p$, $\therefore FQ=(GF+GQ=x+\frac{1}{4}p=$ (by the *Geneſis*) $FC=$(by *Prop.* 6.) FT. *Q. E. D.*

COROLLARY I.

Hence F is the centre of a circle paſſing through Q, C, and T.

COROLLARY II.

The angle formed by the tangent and axis is equal to half the angle formed by the axis, and a right line drawn from the focus to the point of contact; that is, $\angle CTQ=\frac{1}{2}\angle CFQ$, by 31 *Eu.* 3.

PROPOSITION. IX.

IF, from the vertex, a right line be drawn parallel to an ordinate drawn from the point of contact, and cut the tangent, the ſquare of that line ſhall be equal to the rectangle of half the parameter of the axis into half the abſciſſa of that ordinate, that is, $VRq=\frac{1}{2}p\times\frac{1}{2}GV$.

DEMON-

DEMONSTRATION.

Let $VR = b$, the $\triangle$s TVR, TGC are ſimilar, but $VT = \frac{1}{2} GT$, by *Prop.* 5. $\therefore VR = \frac{1}{2} GC$; that is, $b = \frac{1}{2} y$, $\therefore b^2 = (\frac{1}{4} y^2 =$ by *Prop.* 1.) $\frac{1}{2} p \times \frac{1}{2} x$, or $VRq = \frac{1}{2} p \times \frac{1}{2} GV$. *Q. E. D.*

PROPOSITION X.

IF to the tangent drawn to the vertex of any diameter a right line be drawn parallel, the part of that line which lies within the curve, ſhall be biſected by the diameter, that is, the ordinate $xb = bz$. Fig. 6.

DEMONSTRATION.

Produce the diameter Yb, and draw KR, VS parallel to the ordinate FG; then,

1. $\triangle GFT$ or $\square GS : \triangle KzP :: (GFq : Kzq :: GV : KV ::$ by 1 *E.* 6.) $\square GS : \square KS$, $\therefore \triangle KzP = \square KS$.

2. $\triangle GFT$ or $\square GS : \triangle HxP :: (GFq : Hxq :: GV : HV ::) \square GS : \square HS$, $\therefore \triangle HxP = \square HS$; but $\triangle HxP - \triangle KzP = \square HS - \square KS$, *i. e.* the figure $HxzK = \square HR$; from which taking the common figure $HYbzK$, there remains the $\triangle Yxb =$ and ſimilar to the $\triangle bRz$, and conſequently $xb = bz$. *Q. E. D.*

COROLLARY.

The figure $bFTP = \triangle Yxb$, becauſe $\triangle GFT = \square GS$, therefore the figure $HYFT = (\square HF + \triangle GFT = \square HF + \square GS = \square HS =) \triangle HxP$, from which taking the common figure $HYbP$, there remains the figure $bFTP = \triangle Yxb$.

LEMMA.

If FT be $\parallel$ to bp, and the $\triangle brz =$ Trapezium $FbTp$, then $\overline{FT+bp} \times Fb = br \times z$[illegible]because (by Hypotheſis) $\overline{FT+bp} \times p = zb \times q$, $\therefore FT + bp : zb :: q : p ::$ (by ſimilar $\triangle$s) $rb : bF$, and $\overline{FT+bp} \times Fb = zb \times br$. *Q. E. D.* Fig. 11.

DEFINITION.

Let $FS : FO :: 2FT : P$, the parameter belonging to the diameter FY; then, Fig. 6.

PROPOSITION XI.

THE rectangle of the parameter (ſo obtained) into any abſciſſa of that diameter, is equal to the ſquare of the ordinate of that abſciſſa, that is, $P \times Fb = \overline{xb}\rvert^2 = \overline{bz}\rvert^2$. Fig. 6.

DEMONSTRATION.

By the definition $\frac{\mathrm{P}}{2\mathrm{FT}} = \frac{\mathrm{FO}}{\mathrm{FS}} =$ by ſimilar Δs, $\frac{bx}{b\mathrm{Y}}$; and (by the preceding *Lemma*) $2\mathrm{FT} \times \mathrm{F}b = yb \times bx$, $\therefore$ $\left(\frac{\mathrm{P}}{2\mathrm{FT}} \times 2\mathrm{FT} \times b\mathrm{F} = \frac{bx}{by} \times by \times bx\right)$ or $\mathrm{P} \times \mathrm{F}b = \overline{bx}|^2 = \overline{bz}|^2$. *Q. E. D.*

PROPOSITION XII.

Fig. 7. THE parameter of any diameter is equal to the parameter of the axis, added to quadruple the abſciſſa of the ordinate drawn from the vertex of that diameter, *i. e.* $\mathrm{P} = p + 4\mathrm{GV}$.

DEMONSTRATION.

From the vertex draw Vb parallel to the tangent FT, which (by *Prop.* 10.) will be an ordinate to the diameter FY. Then, by reaſon of parallels $b\mathrm{F} = \mathrm{VT} =$(by *Prop.* 5.) $\mathrm{GV} = x$, and by the laſt, $\mathrm{P}x = (b\mathrm{V}q = \mathrm{FT}q = \mathrm{FG}q + \mathrm{GT}q = 4x^2 + y^2 =)\ 4x^2 + px$, $\therefore$ $\mathrm{P} = (4x + p$ or$)\ p + 4\mathrm{GV}$. *Q. E. D.*

PROPOSITION XIII.

THE diſtance from the focus to the vertex of any diameter is equal to one fourth of the parameter of that diameter, that is, $\mathrm{KF} = \frac{1}{4}\mathrm{P}$.

DEMONSTRATION.

By the laſt, $\mathrm{P} = p + 4\mathrm{VT}$, and (by the firſt) $p = 4\mathrm{KV}$, $\therefore$ $\mathrm{P} = 4\mathrm{KV} + 4\mathrm{VT}$; and $\frac{1}{4}\mathrm{P} = (\mathrm{KV} + \mathrm{VT} = \mathrm{KT} =$ by *Prop.* 6.) KF. *Q. E. D.*

PROPOSITION XIV.

IF, from the focus, a perpendicular be drawn to any tangent; then the ſquare of that line ſhall be equal to the rectangle under the focal diſtance, and the diſtance of the point of contact from the focus, *i. e.* $\mathrm{KO}q = \mathrm{KV} \times \mathrm{KF}$.

DEMONSTRATION.

From the vertex draw VO $\parallel$ to GF, which will coincide with the point O, becauſe (by *Prop.* 5.) $\mathrm{GV} = \mathrm{VT}$, $\therefore$ (by 2 *E.* 6.) $\mathrm{TO} = \mathrm{OF}$, and becauſe $\angle\mathrm{KOT}$ is right, $\therefore$ (by the 8 *E.* 6.) $\mathrm{TK} : \mathrm{KO} :: \mathrm{KO} : \mathrm{KV}$, and $\mathrm{KO}q = (\mathrm{TK} \times \mathrm{KV})\ \mathrm{FK} \times \mathrm{KV}$. *Q. E. D.*

PROPO-

PROPOSITION XV.

IF an ordinate to any diameter paſs through the focus, then the abſciſſa of that ordinate ſhall be equal to one fourth, and the ordinate equal to one half of the parameter of that diameter.

DEMONSTRATION.

1. From parallels, $bF=(KT=$by *Prop.* 6. $KF=$by *Prop.* 13.$)\frac{1}{4}P$. Fig. 7.
2. Since $bF=\frac{1}{4}P$, and (by *Prop.* 11.) $P\times bF=bCq$, $\therefore \frac{1}{4}P^2=\overline{bC}|^2$, and $\frac{1}{2}P=bC$. *Q. E. D.*

PROPOSITION XVI.

THE diſtance (in the axis) from the interſection of the tangent, to the end of the ſubnormal, is equal to half the parameter of that diameter, whoſe vertex is the point of contact, that is, $QT=\frac{1}{2}P$.

DEMONSTRATION.

By *Prop.* 13. $FK=\frac{1}{4}P$; and (by *Prop.* 8.) $FK=QK=KT$, $\therefore QT=$ ~~(QT~~ QK $+KT=\frac{1}{4}P+\frac{1}{4}P=)\ \frac{1}{2}P$. *Q. E. D.* ✓

PROPOSITION XVII.

IF a double ordinate be drawn from the point of contact, and another double ordinate be drawn below, and cut the tangent produc'd: Then as the double ordinate paſſing through the point of contact, is to the ſum of the two ordinates, ſo is their difference, to the external part of the lower ordinate added to the difference of the ordinates, that is, $MF : OL :: IL : BL$. Fig. 8.

DEMONSTRATION.

Let $VG=x$, then (by *Prop.* 5.) $GT=2x$, $FG=y$, $OL=c$, $IL=m$, and $LB=d$. Then (by *Prop.* 2.) $p : c :: m : \frac{mc}{p}=LF$; and from ſimilar $\triangle$s, $2x : y :: \frac{mc}{p} : d$, $\therefore 2pxd=ymc$, and (becauſe $px=y^2$) $2yd=mc$, or $2y : c :: m : d$, *i. e.* $MF : OL :: IL : BL$. *Q. E. D.*

PROPOSITION XVIII.

THE ſame things being ſuppos'd as before; the difference of the ordinates is a mean proportional between the double of the

Fig. 8. the upper ordinate, and the external part of the lower, *i. e.* FM : IL :: IL : BI.

Demonstration.

For BI put c, IL, m, and FM, $2y$, then $OL=2y+m$, and (by *Prop.* 17.) $2y : 2y+m :: m : d$, $\therefore d=\frac{2ym+m^2}{2y}$; and $c=(d-m=\frac{2ym+m^2}{2y}-m=)\frac{m^2}{2y}$; that is, FM : IL :: IL : BI. *Q. E. D.*

PROPOSITION XIX.

THE ſame things being ſtill ſuppoſed; as the double of the lower ordinate added to the external part, is to the ſum of the two ordinates, ſo is the external part of the lower ordinate added to the difference of the ordinates, to the difference of the ordinates, that is, OB : LB :: OL : IL.

Demonstration.

Let $OL=c$, $LB=d$, $IL=m$; then $OB=c+d$, and $MF=c-m$. But (by *Prop.* 17.) MF : OL :: IL : BL; that is, $c-m : c :: m : d$, $\therefore cd-dm=cm$; and $cm+dm=cd$; or $c+d : c :: d : m$; that is, OB : LB :: OL : IL. *Q. E. D.*

PROPOSITION XX.

STILL ſuppoſing the ſame things; having OI, and BI given, it is required to find IL.

Let $KL=b$, $IL=m$, $BI=a$, and $OI=c$; then (by *Prop.* 18.) KL : (MF) IL :: IL : BI, that is, $b : m :: m : a$, $\therefore ba=m^2$, and $b=\frac{m^2}{a}$; alſo $c=(b+2m=\frac{m^2}{a}+2m=)\frac{m^2+2am}{a}$, $\therefore ac=m^2+2am$. Whence $m=\sqrt{\overline{a+c}\times a}-a$.

Corollary.

Fig. 8. Hence from a point B without the curve (and not in the ax produc'd) we may draw a tangent. For if, from the given poin we draw IO perpendicular to the axis, and then find a mean proportional between OB and IB, from which if we take IB, and ſet the remainder from I to L; and then from L, draw LF, parall to the axis, the point F is determined, to which, if from the give point B, a right line be drawn, it will touch the curve.

PRO-

PROPOSITION XXI.

IF FP touch the curve in F, and, from any points M, S, in that tangent the right lines BM, SD, be drawn || to the axis, and cut the ordinate in B and D; then MO : FBq : : SR : FDq. Fig. 9.

DEMONSTRATION.

Let MO$=b$, FB$=c$, SR$=d$, and FD$=q$, alſo GV$=$VT$=x$. Then by *Prop.* 11. $x : b : :$ (FTq : FMq : : by ſimiliar Δs) $y^2 : c^2$, and $x : d : :$ (FTq : FSq : : by ſimilar Δs) $y^2 : q^2$, $\therefore$ (by equality) $b : c^2 : : d : q^2$, or MO : FBq : : SR : FDq. *Q. E. D.*

PROPOSITION XXII.

IF, from any point in the tangent, a right line be drawn parallel to the axis, and cut an ordinate, the rectangle of the parameter of the axis into the external part of that line, is equal to the ſquare of the ſegment of the ordinate intercepted between that line and the point of contact; that is, $p\times$MO$=$FBq, or $p\times$RS$=$FDq.

DEMONSTRATION.

By the laſt, $\frac{c^2x}{b} = (y^2=$by *Prop* 1.$)\ px$; alſo $\frac{q^2x}{d} = (y^2=)\ px$, $\therefore$ $\overline{pb}\rvert^2=c^2$, and $dp=q^2$; *i. e.* $p\times$MO$=$FBq and $p\times$RS$=$FDq. *Q.E.D.*

PROPOSITION XXIII.

IF FP touch the Parabola in F, and if, from any point S, in the tangent a right line SD, be drawn parallel to the axis, and cut another right line FC drawn from the point of contact any how within the curve; then the curve ſhall cut the firſt line in the ſame proportion that the firſt line cuts the ſecond; that is, SR:RD::FD: DC. Fig. 9.

DEMONSTRATION.

Draw PC parallel to SD, and let CP$=r$, RS$=c$, FS$=d$, RD $=p$, PS$=m$, FD$=g$, and DC$=b$. Then $c : r : :$ $(d^2 : \overline{d+m}\rvert^2 : :$ by ſimilar Δs$)$ $g^2 : \overline{g+b}\rvert$, and (by ſimilar Δs) $r : g+b : : c+p : g$, therefore $\frac{cg^2+2cgb+cb^2}{g^2} = (r=)\frac{cg+cb+pg+pb}{g}$; and $cgb+cb^2=pg^2+pgb$; and dividing by $g+b$, $cb=pg$, or $c : p : : g : b$; that is, SR : RD : : FD : DC. *Q. E. D.*

PRO-

PROPOSITION XXIV.

Fig. 9. AS the abſciſſa is to the ſquare of the ordinate, ſo is any right line drawn within the curve, and parallel to the axis, to the rectangle of the parts of the ordinate which it divides; that is, VG: FGq::OB:FB×BC.

DEMONSTRATION.

Let $OB=m$, $FB=c$, $BC=r$, $MO=b$, then (by *Prop.* 21) $x:b::y^2:c^2$, and (by the laſt) $b:m::c:r$, $\therefore \frac{c^2x}{y^2}=(b=)\frac{mc}{r}$; and $rcx=my^2$, or $x:y^2::m:rc$; that is, VG:FGq::OB:FB×BC. *Q. E. D.*

COROLLARY.

OB:FB×BC::RD:FD×DC. Becauſe (by this *Prop.*) OB:FB ×BC:: (VG:FGq::) RD:FD×DC.

PROPOSITION XXV.

IF a tangent cut any diameter produced, and if, from the point of contact, an ordinate be drawn to that diameter; then the diſtance (in the diameter produced) between the vertex and interſection of the tangent, ſhall be equal to the abſciſſa of the ordinate, that is, RS =SO. Fig. 10.

DEMONSTRATION.

Let $OS=x$, $Cr=OP=n$, $tr=m$, $RS=a$, $OC=y$, and then Pt, (which is ſuppoſed to be indefinitely near to OC) will be$=y+m$, and $SP=x+n$. Then by ſimilar Δs $m:n::y:x+a$, $\therefore x+a=n\times\frac{y}{m}$, and by *Prop.* 11. $p\times SO=OCq$; alſo $p\times SP=Ptq$, *i. e.* $px=y^2$, and $px+pn=y^2+2ym+m^2$; whence $px=y^2+2ym-pn$, $\therefore y^2=y^2+2ym-pn$, or $2ym=pn$, and $n=\frac{2ym}{p}$; alſo (by ſubſtitution from the firſt equation) $x+a=\left(\frac{2ym}{p}\times\frac{y}{m}=\frac{2y^2}{p}=\frac{2px}{p}=\right)2x$; and $a=2x-x=)\ x$; or, RS=SO. *Q. E. D.*

PROPOSITION XXVI.

IF a diameter be drawn from the interſection of any two tangents, it will biſect the line which joins the points of contact.

DEMON-

DEMONSTRATION.

From the points of contact (Y, C,) draw the ordinates Y *o*, CO; then by the laſt, RS=SO, and RS=S*o*, ∴ SO=S*o*, and conſequently Y *o* and CO being ordinates to the ſame diameter and abſciſſa are equal, and in the ſame right line. *Q. E. D.*

COROLLARY.

Hence we have another method of drawing tangents to the Parabola from any point without the curve. For, if from the given point (as R) you draw a diameter (as RP), and in that diameter ſet, from the vertex, the abſciſſa (SO) equal to the external part (RS), and then through the extremity of the abſciſſa (O) draw a right line parallel to the tangent (xy) at the vertex of that diameter, the extremities of that line (as C, Y,) will be the points in the curve in which lines drawn from the given point will touch it.

PROPOSITION XXVII.

IF, from the extremity of any ordinate (x b) to a diameter, a right line (as x Y*) be drawn at right angles to the diameter; then the diſtance (x* Y*) in that line from the extremity of the ordinate to the diameter, ſhall be a mean proportional between the parameter of the axis, and the abſciſſa of that ordinate; that is, p : x*Y *: : x*Y : F*b*. Fig. 6.

DEMONSTRATION.

Put xY$=a$, Fb=X; then (by *Prop.* 1.) FGq$=px$, and (by *Prop.* 5.) GTq$=4x^2$, ∴ (by 47 *E.* 1.) FTq$=px+4x^2=p+4x\times x$. But (by *Prop.* 12.) xbq$=p+4x\times$X, and from ſimilar Δs FTq : FGq :: xbq : xYq; that is, $\overline{p+4x}\times x : px :: \overline{p+4x}\times X : a^2$, ∴ $a^2=p$X, and $p : a :: a :$ X, or $p : x$Y :: xY : Fb. *Q. E. D.*

Conic Sections.

PART II.

Of the ELLIPSE.

The GENESIS.

Fig. 1. IF, upon a plane, any right line be taken, as AB, and $BH=AK$, and the point G be taken any where in that line; then, if, with the radius AG, from the point K, you deſcribe an arc at F, and with the radius GB from the centre H, you interſect the former arc, and if, from the points H and K, you draw the lines HF, KF, I ſay $HF+KF=AB$. For (by conſtruction) $KF=AG$, and $HF=GB$, $\therefore HF+KF=(AG+GB=)\ AB$. In like manner an indefinite number of points may be found, and if a curve line be drawn through them, it is called an ellipſe.

DEFINITIONS.

1. The points H, and K are called the foci.

Fig. 2. 2. A diameter is a right line which paſſes through C, the middle of AB, and biſects all lines within the curve, that are parallel to the tangent which touches its vertex, and the lines ſo biſected are called ordinates to that diameter; ſo FY, is a diameter, $xo=oz$, are ordinates, being parallel to the tangent which touches the curve in F, the vertex of the diameter.

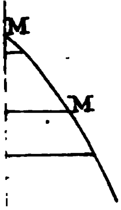
3. The point of concourſe (C) of all the diameters is called the center.

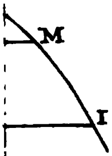
4. That diameter to which the ordinates ſtand at right-angles is called the tranſverſe axis, as AB; and that which paſſes through the center, and cuts it at right angles, is called the conjugate axis, as ED.

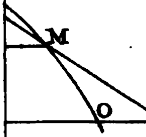
5. The point, where the ordinate interſects the diameter, is called the point of application; as G, and o.

6. The ſegment of the diameter intercepted between the vertex and point of application. is called the abſciſſa; as Fo, oY; or BG, AG.

Parabola







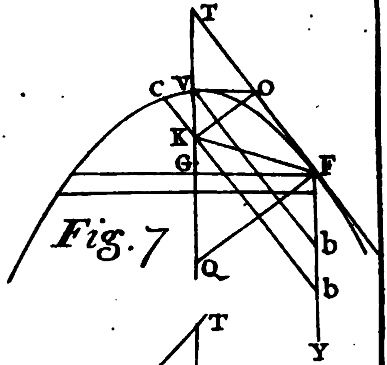


Fig. 7

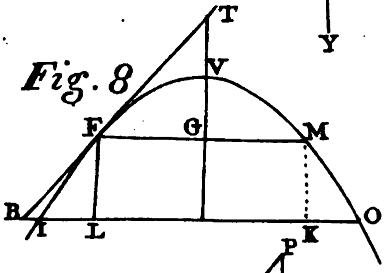


Fig. 8

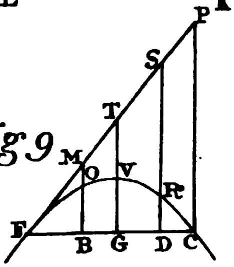
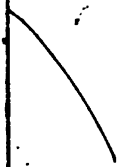


Fig 9



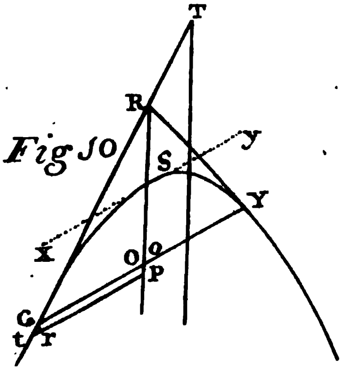
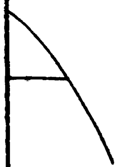


Fig 10



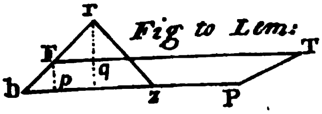


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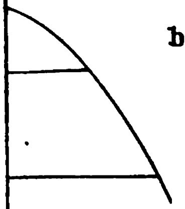
Fig. 11



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PROPOSITION I.

AS the ſquare of any ordinate to the tranſverſe axis, is to the rectangle of the abſciſſas which it divides; ſo is the ſquare of the conjugate, to the ſquare of the tranſverſe axis. Fig. 3.

DEMONSTRATION.

Let $AB=t$, $DE=c$, $KC=b$, $CG=x$, $FG=y$, and $FH=z$; then $KH=2b$, and $KG=b+x$, $GH=b \backsim x$, and *(by the Geneſis)* $KF=t-z$, the points K and H being the foci. Then $KEq-ECq=KCq$; that is, $\frac{1}{4}t^2-\frac{1}{4}c^2=b^2$, by 47. *E.* 1. and (by the 13. and 12. *E.* 2.) $KFq=FHq+KHq\mp\overline{2KH\times HG}$; *i. e.* $t^2-2tz+z^2=z^2+4b^2+4bx-4b^2$; $\therefore z=\frac{t^2-4bx}{2t}$; and by ſquaring both ſides, $\frac{t^4-8t^2bx+16b^2x^2}{4t^2}=$ $(z^2=FG^2+GH^2=)$ $y^2+b^2-2bx+x^2$; which being clear'd of fractions and contradictory terms will be $t^4+16b^2x^2=4t^2y^2+4t^2b^2+4t^2x^2$; and ſubſtituting, for $16b^2$ and $4b^3$ in this equation, their reſpective values in the firſt, and throwing out contradictory terms, and dividing by 4, we ſhall have $t^2y^2=\frac{1}{4}t^2c^2-c^2x^2$; which reduced to an analogy gives $y^2:\overline{\frac{1}{2}t+x}\times\overline{\frac{1}{2}t-x}::c^2:t^2$; *i. e.* $FGq:AG\times GB::DE^2:AB^2$. *Q. E. D.*

COROLLARY.

Let any abſciſſa be x, and its ordinate y, the tranſverſe axis t, and the conjugate c (which ſymbols repreſent the ſame things in all the following demonſtrations) then by this *Theorem*, $t^2:c^2::\overline{t-x}\times x:y^2$, or $t^2y^2=c^2tx-c^2x^2$; which I call the equation of the curve.

Definition. A third proportional to the tranſverſe and conjugate axes is called the parameter of the axis; that is, if you put p for the parameter, then $t:c::c:p$, $\therefore tp=c^2$.

PROPOSITION II.

AS the tranſverſe axis, is to its parameter; ſo is the rectangle of any two abſciſſas, to the ſquare of the ordinate which divides them.

DEMONSTRATION.

By the conſtruction of the parameter $tp=c^2$, and by putting tp in the equation of the curve for c^2, we ſhall have a new equation of the curve in the terms of the parameter, *&c. viz.* $ty^2=tpx-px^2$, or $y^2=\frac{p}{t}\times\overline{tx-x^2}$, $\therefore t:p::\overline{t-x}\times x:y^2$. *Q. E. D.*

COROLLARY.

As the rectangle of any two abſciſſas, is to the ſquare of the ordinate which divides them; ſo is the rectangle of any other two abſciſſas, to the ſquare of the ordinate which divides them: for, (by this *Prop.*) $\overline{t-x}\times x:y^2::(t:p::)\ \overline{t-X}\times X:Y^2$.

PROPOSITION III.

THE tranſverſe axis into one fourth of its parameter, is equal to the rectangle of the greateſt and leaſt diſtance of either focus from the vertex; that is, $\frac{1}{4}p\times AB=AH\times HB=BK\times KA$.

DEMONSTRATION.

Fig. 3. Let $HB=q$, then $HA=t-q$, and $CH=\frac{1}{2}t-q$. But HEq= ECq+CHq. *i. e.* $\frac{1}{4}t^2=\frac{1}{4}t^2-tq+q^2+\frac{1}{4}c^2$, or $\overline{t-q}\times q=(\frac{1}{4}c^2=)\ \frac{1}{4}tp$; or $\frac{1}{4}p\times AB=AH\times HB$. *Q. E. D.*

COROLLARY.

The ſemi-conjugate axis is a mean proportional between the greateſt and leaſt diſtance of either focus from the vertices: for ſince $\overline{t-q}\times q=\frac{1}{4}c^2$, $\therefore t-q:\frac{1}{2}c::\frac{1}{2}c:q$; that is, AH:CD::CD:HB.

PROPOSITION IV.

THE parameter of the axis is double the ordinate applied to the focus.

DEMONSTRATION.

Fig. 4. Let the focal diſtance be q, and the ordinate paſſing through the focus y; then (by the 2.) $t:p::\overline{t-q}\times q:y^2$. But (by *Prop.* 3.) $\overline{t-q}\times q=\frac{1}{4}tp$, $\therefore t:p::\frac{1}{4}tp:(\frac{1}{4}p^2=)\ y^2$, and $\frac{1}{2}p=y$, or $p=2y$. *Q. E. D.*

PROPOSITION V.

THE diſtance between the foci is a mean proportional between the ſum and difference of the tranſverſe and conjugate axis; *i. e.* AB+DE:KH::KH:AB−DE.

DEMONSTRATION.

For KH, put b; then KDq−CDq=KCq; *i. e.* $\frac{1}{4}t^2-\frac{1}{4}c^2=\frac{1}{4}b^2$,

$\frac{1}{4}b^2$, or $t^2-c^2=b^2$, $\therefore t+c:b::b:t-c$; or AB+DE:KH::KH: AB−DE. *Q. E. D.*

PROPOSITION VI.

A Fourth proportional to the conjugate, tranſverſe, and any ordinate, is equal to a mean proportional between the abſciſſas of that ordinate.

DEMONSTRATION.

Let the fourth proportional be b; then $c:t::y:b$, $\therefore \frac{ty}{c}=b$; but (by *Prop.* 1.) $t^2:c^2::\overline{t-x}\times x:y^2$, $\therefore$ (by 22. *E.* 6.) $t:c::\sqrt{\overline{t-x}\times x}:y$, and $\sqrt{\overline{t-x}\times x}=\left(\frac{ty}{c}=\right) b$. *Q. E. D.* Fig. 4.

PROPOSITION VII.

THE diſtance between the foci, is a mean proportional between the tranſverſe axis, and the difference of the tranſverſe axis and the parameter; *i. e.* AB:KH::KH:AB−LR.

DEMONSTRATION.

KDq−CDq=KCq; that is, $\frac{1}{4}t^2-\frac{1}{4}c^2=\frac{1}{4}b^2$, or $t^2-c^2=b^2$; but $tp=c^2$, $\therefore t^2-tp=b^2$, and $t:b::b:t-p$, or AB:KH::KH: AB−LR. *Q. E. D.*

PROPOSITION VIII.

AS the ſquare of any ordinate, is to the rectangle of its abſciſſas; ſo is the ſquare of the conjugate, to the ſquare of the conjugate added to the ſquare of the diſtance of the foci; that is, FGq: AG×GB::EDq:EDq+KHq.

DEMONSTRATION.

KEq=KCq+CEq; *i. e.* $\frac{1}{4}t^2=\frac{1}{4}b^2+\frac{1}{4}c^2$, or $t^2=b^2+c^2$; but (by *Prop.* 1.) $y^2:\overline{t-x}\times x::c^2:(t^2=)\,b^2+c^2$; or FGq:AG×GB ::EDq:EDq+KHq. *Q. E. D.*

PROPOSITION IX.

AS the ſquare of any ordinate, is to the rectangle of its abſciſſa into the parameter; ſo is the difference between the ſquare of the conjugate axis, and the rectangle of the abſciſſa into the parameter, to the ſquare of the conjugate axis; *i. e.* $FGq : BG \times LR :: EDq - \overline{BG \times LR} : EDq$.

DEMONSTRATION.

Fig. 4. By the equation of the curve, $t^2 y^2 = c^2 t x - c^2 x^{t}$. But $\frac{c^2}{p} = t$, $\therefore$ (by ſubſtitution) $\frac{c^4 y^2}{p^2} = \frac{c^4 x}{p} - c^2 x^2$, and $c^2 y^2 = c^2 p x - p^2 x^2$; *i. e.* $y^2 : p x :: c^2 - p x : c^2$; or $FGq : BG \times LR :: EDq - \overline{BG \times LR} : EDq$. *Q. E. D.*

PROPOSITION X.

AS the ſquare of the conjugate axis, is to the ſquare of the tranſverſe axis; ſo is the rectangle of any two abſciſſas of the conjugate axis, to the ſquare of the ordinate which divides them; *i. e.* $DEq : ABq :: Dh \times hE : Fhq$.

DEMONSTRATION.

Let $Eh = x$, and $Fh = y$; then (by *Prop.* 1.) $ABq : EDq :: AG \times GBq : FGq$. But (by 5. *E.* 2.) $AG \times GB = \overline{CB}^2 - \overline{Fh}^2$, and $\overline{Ch}^2 = (FGq =) CEq - \overline{Dh \times hE}$, $\therefore$ (by ſubſtitution) $ABq : EDq :: CBq - Fhq : CEq - \overline{Dh \times hE}$; that is, $t^2 : c^2 :: \frac{1}{4} t^2 - y^2 : \frac{1}{4} c^2 - c x + x^2$; which reduced to an equation produces $c^2 y^2 = t^2 c x - t^2 x^2$; *i. e.* $c^2 : t^2 :: \overline{c - x} \times x : y^2$; or $DEq : ABq :: Dh \times hE : Fhq$. *Q. E. D.*

DEFINITION.

A third proportional to the conjugate and tranſverſe axis, is a parameter to the conjugate axis; that is, p being put for the parameter, $c : t :: t : p$, $\therefore cp = t^2$.

PROPOSITION XI.

AS the conjugate axis is to its parameter, ſo is the rectangle of any two abſciſſas of the conjugate axis, to the ſquare of the ordinate which divides them.

DEMONSTRATION.

For t^3, in the laſt equation, put its equal cp; then $c y^2 = c p x - p x^2$; *i. e.* $c : p :: \overline{c - x} \times x : y^2$. *Q. E. D.*

PROPOSITION XII.

AS the ſquare of any ordinate of the conjugate, is to the rectangle of the abſciſſas which it divides; ſo is the ſum of the ſquares of the diſtance of the foci and the conjugate axis, to the ſquare of the conjugate axis.

DEMONSTRATION.

By the 10. $y^2 : cx - x^2 :: t^2 : c^2$, and (by 47. *E.* 1.) $t^2 = b^2 + c^2$, Fig. 4. $\therefore$ (by ſubſtitution $y^2 : cx - x^2 :: b^2 + c^2 : c^2$; that is, F h q : D h × h E :: K H q + E D q : E D q. *Q. E. D.*

PROPOSITION XIII.

IN any tangent to the Ellipſe, if, from the point of contact, an ordinate be drawn to the axis, and the tangent continued to cut the axis produced; then it will hold, as the diſtance (in the axis) between the center and ordinate, is to the abſciſſa of that ordinate; ſo is the remainder of the axis, to (the diſtance between the ordinate and interſection of the tangent with the axis, that is) the *Subtangent*; viz. C G : G B : : A G : G T.

DEMONSTRATION.

Let F p be an indefinitely ſmall part of the curve, and continued to cut the axis produced in T; draw the ordinate F G, and, parallel Fig. 5. to it, p q, draw alſo F r parallel to the axis, and, for F r, put n, p r, m, and B T, a. Then is B q $= x + n$, A q $= t - x - n$, p q $= y + m$, and G T $= a + x$. But (by ſimilar Δs) p r : r F : : F G : G T; *i. e.* $m : n :: y : x + a$, $\therefore n \times \frac{y}{m} = x + a$, and (by *Prop.* 2.) $t : p :: tx - x^2 + tn - 2xn - n^2 : y^2 + 2ym + m^2$. Alſo, $t : p :: tx - x^2 : y^2$, $\therefore ptx - px^2 + ptn - 2pxn = ty^2 + 2tym$, and $ty^2 = ptx - px^2$, $\therefore ptx - px^2 + ptn - 2xnp - 2tym =$ $(ty^2 =)$ $ptx - px^2$, or $ptn - 2nxp = 2tym$, and $n = \frac{2tym}{pt - 2px}$, $\therefore x + a =$ $\left(n \times \frac{y}{m} = \frac{2tym}{pt - 2px} \times \frac{y}{m} = \frac{ty^2}{p} \times \frac{2}{t - 2x} =\right.$ (becauſe, by the 2d. $tx - x^2 = \frac{y^2 t}{p}$) $\left.\frac{2tx - 2x^2}{t - 2x} =\right)$ $\frac{tx - x^2}{\frac{1}{2}t - x}$, $\therefore \frac{1}{2}t - x : x :: t - x : x + a$; or C G : G B : : A G : G T. *Q. E. D.*

PROPOSITION XIV.

AS the diſtance from the center to the ordinate drawn from the point of contact, is to half the tranſverſe axis; ſo is half the tranſverſe axis, to the diſtance from the center to the concurring of the tangent with the axis produc'd; *i. e.* CG : CB : : CB : CT.

DEMONSTRATION.

CT=CG+GT; but $CT=\frac{1}{2}t+a$, $CG=\frac{1}{2}t-x$, and (by *Prop.* 13.) $GT=\frac{tx-x^2}{\frac{1}{2}t-x}$, $\therefore \frac{1}{2}t+a=(\frac{1}{2}t-x+\frac{tx-x^2}{\frac{1}{2}t-x}=)\frac{\frac{1}{4}t^2}{\frac{1}{2}t-x}$; that is, $\frac{1}{2}t-x:\frac{1}{2}t::\frac{1}{2}t:\frac{1}{2}t+a$; or CG : CB : : CB : CT. *Q. E. D.*

PROPOSITION XV.

AS the diſtance from the center to the ordinate drawn from the point of contact, is to half the tranſverſe; ſo is the abſciſſa of that ordinate, to the external part of the tranſverſe; that is, CG : CB : : GB : BT.

DEMONSTRATION.

By the 14. $\frac{1}{2}t+a=\frac{\frac{1}{4}t^2}{\frac{1}{2}t-x}$, $\therefore \frac{1}{4}t^2+\frac{1}{2}ta-\frac{1}{2}tx-xa=\frac{1}{4}t^2$, and $a=\frac{\frac{1}{2}tx}{\frac{1}{2}t-x}$; *i. e.* $\frac{1}{2}tx-x:\frac{1}{2}t::x:a$; or CG : CB : : GB : BT.

PROPOSITION XVI.

AS the diſtance from the center to the ordinate drawn from the point of contact, is to half the tranſverſe; ſo is the greater abſciſſa of that ordinate, to the tranſverſe axis, added to the external part; that is, CG : CB : : AG : AT.

DEMONSTRATION.

Fig. 5. By *Prop.* 15, $a=\frac{\frac{1}{2}tx}{\frac{1}{2}t-x}$, $\therefore t+a=(t+\frac{\frac{1}{2}tx}{\frac{1}{2}t-x}=)\frac{\frac{1}{2}t^2-\frac{1}{2}tx}{\frac{1}{2}t-x}$; *i. e.* $\frac{1}{2}t-x:\frac{1}{2}t::t-x:t+a$, or CG : CB : : AG : AT. *Q. E. D.*

PROPOSITION XVII.

AS the greater abſciſſa of the ordinate drawn from the point of contact, is to the ſum of the tranſverſe and external part; ſo is the leſs abſciſſa of that ordinate, to the external part; that is, AG : AT : : GB : BT.

DEMONSTRATION.

By *Prop.* 15. $\frac{1}{2}t-x:\frac{1}{2}t::x:a$; and (by the 16.) $\frac{1}{2}t-x:\frac{1}{2}t::t-x:t+a$, ∴ (by equality) $t-x:t+a::x:a$; or AG : AT : : GB : BT.

PRO-

PROPOSITION XVIII.

AS the diſtance from the center to the concurring of the tangent, is to half the tranſverſe; ſo is the external part, to the abſciſſa of the ordinate from the point of contact; that is, CT : CB : : BT : BG.

DEMONSTRATION.

By *Prop.* 15. $\frac{1}{2}ta=\frac{1}{2}tx+xa$, $\because$ $x=\frac{\frac{1}{2}ta}{\frac{1}{2}t+a}$, and $\frac{1}{2}t+a : \frac{1}{2}t :: a : x$; or, CT : CB : : BT : BG.

PROPOSITION XIX.

AS half the tranſverſe added to the external part, is to the tranſverſe added to the external part; ſo is the external part, to the ſubtangent; that is, CT : AT : : BT : GT.

DEMONSTRATION.

By *Prop.* 18. $x=\frac{\frac{1}{2}ta}{\frac{1}{2}t+a}$, $\therefore$ $x+a=(a+\frac{\frac{1}{2}ta}{\frac{1}{2}t+a}=)\frac{ta+a^2}{\frac{1}{2}t+a}$; and, $\frac{1}{2}t+a : t+a :: a : x+a$, or, CT : AT : : BT : : GT. *Q. E. D.*

PROPOSITION XX.

AS the greater abſciſſa of the ordinate drawn from the point of contact, is to half the tranſverſe; ſo is the ſubtangent, to the external part; *i. e.* AG : CB : : GT : BT. Fig. 5.

DEMONSTRATION.

By *Prop.* 15. $\frac{1}{2}t-x=\frac{\frac{1}{2}tx}{a}$, $\therefore$ $t-x=(\frac{1}{2}t+\frac{\frac{1}{2}tx}{a}=)\frac{\frac{1}{2}ta+\frac{1}{2}tx}{a}$, and $t-x : \frac{1}{2}t :: x+a : a$, or AG : CB : : GT : BT. *Q. E. D.*

PROPOSITION XXI.

AS the tranſverſe added to the external part, is to half the tranſverſe; ſo is the ſubtangent, to the abſciſſa; *i. e.* AT : CB : : GT : GB.

DEMONSTRATION.

By *Prop.* 18. $\frac{1}{2}t+a=\frac{\frac{1}{2}ta}{x}$, $\therefore$ $t+a=(\frac{1}{2}t+\frac{\frac{1}{2}ta}{x}=)\frac{\frac{1}{2}tx+\frac{1}{2}ta}{x}$, and $t+a : \frac{1}{2}t :: x+a : x$, or AT : CB : : GT : GB. *Q. E. D.*

PROPO-

PROPOSITION XXII.

THE ordinate drawn from the point of contact, divided by the ſubtangent, is equal to the quotient of the diſtance between the center and that ordinate, divided by that ordinate multiplied by the parameter divided by the tranſverſe axis; that is, $\frac{GF}{GT}=\frac{CG}{GF}\times\frac{p}{t}$

DEMONSTRATION.

By *Prop.* 13. $tx-x^2=\overline{\tfrac{1}{2}t-x}\times\overline{x+a}$; and (by *Prop.* 2.) $t:p::(tx-x^2)\ \overline{\tfrac{1}{2}t-x}\times\overline{x+a}:y^2$, $\therefore ty^2=p\times\overline{\tfrac{1}{2}t-x}\times\overline{x+a}$, and if you divide by $x+a$, $\frac{ty^2}{x+a}=p\times\overline{\tfrac{1}{2}t-x}$; and again if you divide by ty, $\frac{y}{x+a}=\left(\frac{p\times\overline{\frac{1}{2}t-x}}{ty}=\right)\frac{\frac{1}{2}t-x}{y}\times\frac{p}{t}$. *Q. E. D.*

PROPOSITION XXIII.

Fig. 6. IF perpendiculars be drawn from the ends of the tranſverſe, and from the center, ſo as to cut any tangent, and alſo if from the point of contact be drawn an ordinate, theſe four lines ſhall be proportional; that is, AO : CP :: GF : BQ.

DEMONSTRATION.

By *Prop.* 19. TA : TC :: TG : TB, ∴ (by 4 *Eu.* 6.) AO : CP :: GF : BQ. *Q. E. D.*

COROLLARY.

AO × BQ = CP × GF.

PROPOSITION XXIV.

IF perpendiculars be drawn from the extremities of the tranſverſe, and cut any tangent, then the rectangle of theſe perpendiculars ſhall be equal to the rectangle of the greateſt and leaſt diſtance of either of the foci from the vertices; *i. e.* AO × BQ = AH × HB = BK × KA.

DEMONSTRATION.

Let BQ $=m$, AO $=n$, and AK = HB $=q$; then (by ſimilar Δs) $m:y::(a:x+a::$ by *Prop.* 20.) $\tfrac{1}{2}t:t-x$; and $n:y::(t+a:x+a::$ by *Prop.* 21.) $\tfrac{1}{2}t:x$, $\therefore m=\frac{\frac{1}{2}ty}{t-x}$, and $n=\frac{\frac{1}{2}ty}{x}$, and (by multiplicatiplication,) $mn = \frac{\frac{1}{4}t^2y^2}{\overline{t-x} \times x}$, ∴ $mn : \frac{1}{4}t^2 :: (y^2 : \overline{t-x} \times x ::$ by *Prop.* 2. $p : t ::) \frac{1}{4}pt : \frac{1}{4}t^2$; and $mn = (\frac{1}{4}pt =$ by *Prop.* 3.) $\overline{t-q} \times q$, or AO ×BQ=AH×HB=BK×KA. *Q. E. D.*

LEMMA.

If, from the ends of the chord AB, the perpendiculars AD, BC, Fig. 7. be drawn to meet the circle, then right lines connecting A and C, B and D, ſhall be diameters, and conſequently the point of their concourſe, O, will be the center of the circle, through which if a right line be drawn any how, it will make the alternate ſegments of the perpendiculars equal.

DEMONSTRATION.

By hypotheſis the angles A and B are right, ∴ (by 31 *E.* 3.) AC and BD are diameters, and O the center; but △OPD is ſimilar to △OQB, ∴ OB : BQ :: OD : DP; but OB=OD, ∴ BQ=PD. *Q. E. D.*

PROPOSITION XXV.

IF from the interſections (P, S) of a circle, whoſe diameter is the Fig. 8. tranſverſe axis with any tangent, perpendiculars P*k*, S*h*, be drawn, they ſhall cut the tranſverſe axis in the focal points; that is, the points *k*, *h* coincide with K, H.

DEMONSTRATION.

The △s TBQ, ATO are ſimilar to the △s TS*h*, TP*k*, each having a right angle, and the angle T common, ∴ AO : P*k* :: S*h* : BQ, and AO×BQ=(P*k*×S*h*= by the precedent *Lemma* P*k* ×*kt*; or *hr*×S*h*=by 35 *E.* 3.) AK×*k*B or B*h*×*h*A; but AO× BQ=A*k*×KB or AH×HB, by *Prop.* 24., ∴ the points H, *h* and K, *k* are coincident. *Q. E. D.*

COROLLARY.

PK×SH=$\frac{1}{4}pt$; becauſe PK×SH=(AK×KB=$\overline{t-q} \times q$= by *Prop.* 3.) $\frac{1}{4}tp$.

PROPOSITION XXVI.

IF to any point of the curve, right lines be drawn from the foci, and one of the lines be continued; then a right line biſecting the external angle, ſhall touch the curve in the angular point.

DEMONSTRATION.

Fig. 9. Take $FX=FH$; then (becauſe by the hypotheſis $\angle XFT=\angle TFH$) if you take any point S, in the line FT, $HS=XS$, by 4 *E.* 1. Draw KS, then $KS+(SX=)$ SH is greater than $(KH=)$ AB, and ∴ the point S is without the curve, for if it were in the curve, $KS+SH$ (by the *Geneſis* $=$) AB.

PROPOSITION XXVII.

LINES drawn from the foci to the point of contact, make equal angles with the tangent.

DEMONSTRATION.

By *Prop.* 26. $\angle HFT=(\angle XFT=$by 15 *E.* 1.) $\angle KFO$. *Q.E.D.*

PROPOSITION XXVIII.

A Right line perpendicular to the tangent, at the point of contact, biſects the angle form'd by lines drawn from the foci to the ſame point; that is, if FY be perpendicular to OT, then $\angle KFY=\angle HFY$.

DEMONSTRATION.

Fig. 9. The $\angle PFY=\angle YFT$ by hypotheſis, from which if you take away the $\angle KFP=\angle HFS$, by *Prop.* 27. there remains $\angle KFY=\angle YFH$. *Q.E.D.*

PROPOSITION XXIX.

IF, on the tangent at the point of contact, a perpendicular be drawn, and cut the axis, it will divide the diſtance between the foci, in the ſame proportion, as lines drawn from the foci to the ſame point; *i.e.* $KF:FH::KY:YH$.

DEMONSTRATION.

In the $\triangle KFH$, the $\angle KFY=\angle YFH$ by *Prop.* 28. ∴ (by the 3d *E.* 6.) $KF:FH::KY:YH$. *Q.E.D.*

PROPOSITION. XXX.

IF, on the tangent, at the point of contact, a perpendicular be drawn, and if, from the point, where that perpendicular cuts the axis, lines be drawn perpendicular to lines drawn from the foci to the point of contact, then the diſtance on theſe lines, from the point of contact to the perpendiculars, will be equal to half the parameter of the axis; that is, $Fq=Fr=\frac{1}{2}p$.

DEMONSTRATION.

From the points S, P, where a circle on the tranſverſe cuts the tangent, draw the lines SH, PK to the foci, which will be perpendicular to PT, by *Prop.* 25. and conſequently parallel to FY; continue KF, HS, till they concur in X; then $KX=t$, and $HX=2HS$ by *Prop.* 26. and becauſe $\triangle KFY$ is ſimilar $\triangle KXH$, and $\triangle KPF$, ſimilar $\triangle YFq$, $\therefore KX:XH::(KF:FY::)\ KP:Fq$; and $KX\times Fq=XH\times KP$; and $\frac{1}{2}KX\times Fq=\frac{1}{2}XH\times KP$; that is, $\frac{1}{2}t\times Fq=(SH\times KP=$ by *Prop.* 25.) $\frac{1}{4}pt$, $\therefore Fq=\frac{1}{2}p$; but (by *Prop.* 28.) $\angle YFq=YFr$, and (by 26. *E.* 1.) $Fr=Fq$, $\therefore Fq=Fr=\frac{1}{2}p$. *Q. E. D.* Fig. 9.

PROPOSITION XXXI.

IF perpendiculars from the vertices cut any tangent, then the part of the tangent, intercepted between the interſections, ſhall be the diameter of a circle, whoſe circumference ſhall paſs thro' the foci.

DEMONSTRATION.

By *Prop.* 24. $AO\times BQ=AK\times KB$, $\therefore AO:AK::KB:BQ$; but the angles OAK, and QBK are right, $\therefore$ by 6. *E.* 6. the $\triangle$s OAK and BQK are ſimilar, and $\angle AOK=\angle QKB$. But $\angle AKO+\angle AOK=L$, $\therefore AKO+QKB=L$, and conſequently (by 13 *E.* 1.) $\angle OKQ=L$, and (by 31. *E.* 3.) OQ is a diameter of a circle, whoſe periphery will paſs through K. In like manner the $\angle OHQ$ is proved a right angle. *Q. E. D.* Fig. 10.

COROLLARY.

If OQ be biſected in N, then $NO=NQ=NK=NH$.

PROPOSITION XXXII.

IF, from either focus, a right line be drawn through the point of contact, and continued till it be equal to the tranſverſe axis, and the extremity connected by a ſtreight line to the other focus; then the diſtance between the center, and the interſection of the laſt line with the tangent, is equal to half the tranſverſe axis; that is, $CS=CB$. Fig. 11.

DEMONSTRATION.

In the $\triangle$s HCS and HKX, the $\angle KHX$ is common, and $KC=CH$; alſo $HS=SX$ by *Prop.* 26, $\therefore$ (by 6. *E.* 6.) the $\triangle$s are ſimilar, and CS is parallel to KX; alſo $CS = (\frac{1}{2}XK = \frac{1}{2}AB =) CB$. *Q. E. D.*

PROPOSITION XXXIII.

IF, from the focus, a right line be drawn to the point of contact, and another through the center parallel to the tangent, then the diſtance between the point of contact, and the interſection of theſe lines, is equal to half the tranſverſe axis.

DEMONSTRATION.

Draw CS parallel to KF; then is the figure ZCSF a parallelogram, and $ZF = (CS =$ by *Prop.* 32.$)$ BC. *Q. E. D.*

PROPOSITION XXXIV.

Fig. 12. IF to the tangent, drawn to the vertex of any diameter, a right line be drawn parallel, the part of that line which lies within the curve ſhall be biſected by the diameter; that is, $xb = bz$, alſo $\triangle Vxp + \triangle CVo = \triangle BCS = \triangle dpz + \triangle Cdr$.

DEMONSTRATION.

Let $dz = y$, $dp = c$, $Cd = n$, $BS = r$, $dr = p$, $Vx = Y$, $CV = g$, $Vo = q$, $Vp = b$, $Bd = x$, and $BV = X$. Then, 1. (from ſimilar $\triangle$s) Fig. 12. $\frac{y}{c} = \left(\frac{FG}{GT} = \right.$ by *Prop.* 22.$\left.\right) \frac{CG}{GF} \times \frac{p}{t}$. But $\frac{CG}{GF} = \frac{\frac{1}{2}t}{r}$, $\therefore \frac{y}{c} = \frac{\frac{1}{2}t}{r} \times \frac{p}{t}$; divide by $\frac{p}{t}$, and $\frac{\frac{1}{2}t}{r} = \frac{y}{c} \times \frac{t}{p}$; which multiplied by cry, gives $\frac{1}{2}tcy = ry^2 \times \frac{t}{p}$. But $y^2 \times \frac{t}{p} = \overline{t-x} \times x$ by *Prop.* 2, $\therefore \frac{1}{2}tcy = r \times \overline{t-x} \times x$, and $\frac{1}{2}tcy + rn^2 = (r \times \overline{t-x} \times x + rn^2 =) r \times \overline{\overline{t-x} \times x + n^2}$; and (by 5 *E.* 2.) $\overline{\overline{t-x} \times x} + n^2 = \frac{1}{4}t^2$, $\therefore \frac{1}{2}tcy + rn^2 = r \times \frac{1}{4}t^2$. Divide by $\frac{1}{2}t$, and $cy + \frac{rn^2}{\frac{1}{2}t} = r \times \frac{1}{2}t$; by ſimilar $\triangle$s, $\frac{1}{2}t : r :: n : p$, $\therefore \frac{rn}{\frac{1}{2}t} = p$, $\therefore$ (by ſubſtitution) $cy + pn = r \times \frac{1}{2}t$, or $dp \times dz + dr \times Cd = BS \times BC$; that is, $\triangle BSC = \triangle dpz + Cdr$.

2. By ſimilar $\triangle$s $\frac{Y}{b} = \left(\frac{FG}{GT} = \right.$ by *Prop.* 22.$\left.\right) \frac{CG}{GF} \times \frac{p}{t}$; but $\frac{CG}{GF} = \frac{g}{q}$, $\therefore \frac{Y}{b} = \frac{g}{q} \times \frac{p}{t}$; divide by $\frac{p}{t}$, and $\frac{g}{q} = \frac{Y}{b} \times \frac{t}{p}$, and multiplying by bqY, $gbY = qY^2 \times \frac{t}{p}$. But $tX - X^2 = Y^2 \times \frac{t}{p}$, $\therefore gbY = q \times \overline{tX - X^2}$,

and $gbY+qg^2=\overline{g\times tX-X^2}+g^2$; but (by 5. *E.* 2.) $tX-X^2+g^2=\frac{1}{4}t^2$, therefore $gbY+g^2q=q\times\frac{1}{4}t^2$; divide by g, and $bY+qg=\frac{q\times\frac{1}{4}t^2}{g}$; but $g:q::\frac{1}{2}t:r$, $\therefore\frac{\frac{1}{2}tq}{g}=r$; and (by ſubſtitution) $bY+qg=r\times\frac{1}{2}t$, that is, $Vp\times Vx+Vo\times CV=CB\times BS$, or $\triangle Vxp+\triangle CVo=$ ($\triangle BCS=$ by the 1ſt part) $\triangle dpz+\triangle Cdr$.

3. From both ſides of the laſt equation take $\triangle pbC$, and there remains $\triangle bzr=$ and ſimilar to $\triangle obx$, $\therefore xb=bz$. *Q. E. D.*

PROPOSITION XXXV.

THE ſame things being ſuppos'd as in the laſt, the $\triangle BSC=\triangle CFT$; alſo the *Trapezium* $dBSr=\triangle pdz$; the *Trapezium* $FbpT=\triangle bzr$; and $\overline{FT+bp}\times bF=zb\times br$.

DEMONSTRATION.

From ſimilar $\triangle$s $BS:FG::$ ($BC:GC::$ by *Prop.* 14.) $CT:BC$, Fig. 12. $\therefore BS\times BC=FG\times CT$, or $\triangle BSC=\triangle CFT=$ (by *Prop.* 34.) $\triangle pdz+\triangle Cdr$; from the 1ſt. and 3d. Equ. take $\triangle Cdr$, and there remains the *Trapezium* $dBSr=\triangle pdz$; alſo from the 2d and 3d take the $\triangle pbC$, and there remains the *Trapezium* $FbpT=\triangle bzr$, $\therefore$ (by the *Lemma* to *Prop* 11. of the *Parabola*) $\overline{FT+pb}\times bF=zb\times br$. *Q. E. D.*

LEMMA.

The ſame things being ſtill ſuppoſed, $Yb\times nT=zb\times br$; for Fig. 12. $YC=(FC):bC:FT:bp$, and by compoſition $YC+bC:bC::FT+bp:bp$; and (by alternation $YC+bC=$) $Yb:FT+bp::$ ($bC:bp::$) $np=bF:nT$, $\therefore Yb\times nT=$ ($\overline{FT+bp}\times bF=$ by *Lemma* to *Prop.* 11. Part 1.) $zb\times br$.

DEFINITION.

If ($FS:FQ::$) $br:bz::2FT:P$, the parameter belonging to the diameter FY, then $P=bz\times\frac{2FT}{br}$, and,

PROPOSITION XXXVI.

AS any diameter is to its parameter (ſo obtained) ſo is the rectangle of any abſciſſas of that diameter, to the ſquare of the ordinate which divides them; that is, (if you put D for the diameter FY, x for the abſciſſa Fb, and y for the ordinate $bz=bx$, then) $D:P::\overline{D-x}\times x:y^2$.

DEMONSTRATION.

By the *Definition*, $\mathrm{P}=y\times\frac{2\mathrm{FT}}{\mathrm{br}}$, $\therefore\frac{\mathrm{P}\times\overline{\mathrm{D}-x}\times x}{\mathrm{D}}=(\frac{y\times 2\mathrm{FT}}{\mathrm{br}}\times\frac{\overline{\mathrm{D}-x}\times x}{\mathrm{D}}=)\frac{y}{\mathrm{br}}\times\overline{\mathrm{D}-x}\times x\times\frac{2\mathrm{FT}}{\mathrm{D}}$: but $\frac{2\mathrm{FT}}{\mathrm{D}}=(\frac{\mathrm{FT}}{\frac{1}{2}\mathrm{D}}=$ by ſimilar Δs) $\frac{\mathrm{Tn}}{(\mathrm{np}=)\,x}$, and (by ſubſtitution) $\mathrm{p}\times\frac{\overline{\mathrm{D}-x}\times x}{\mathrm{D}}=(\frac{y}{\mathrm{br}}\times\overline{\mathrm{D}-x}\times\mathrm{Tn}=$ (by the preceding *Lemma*) $\frac{y}{\mathrm{br}}\times y\times\mathrm{br}=)\,y^2$, and $\therefore\mathrm{D}:\mathrm{P}::\overline{\mathrm{D}-x}\times x:y^2$. *Q. E. D.*

PROPOSITION XXXVII.

As any parameter is to its diameter, ſo is the ſquare of its conjugate, to the ſquare of the diameter; *i. e.* $\mathrm{P}:\mathrm{YF}::\mathrm{FYq}:\mathrm{XDq}$.

DEMONSTRATION.

In this caſe $x=\frac{1}{2}\mathrm{D}$, and $y=\frac{1}{2}\mathrm{C}$, $\therefore$ by the laſt, $\mathrm{D}:\mathrm{P}::(\frac{1}{4}\mathrm{D}^2:\frac{1}{4}\mathrm{C}^2::)\,\mathrm{D}^2:\mathrm{C}^2$, or $\mathrm{P}:\mathrm{YF}::\mathrm{FYq}:\mathrm{XDq}$.

COROLLARY.

Hence any conjugate diameter is a mean proportional between the diameter to which it is conjugate, and the parameter of that diameter; for by this *Prop.* $\mathrm{DP}=\mathrm{C}^2$, $\therefore\mathrm{D}:\mathrm{C}::\mathrm{C}:\mathrm{P}$.

PROPOSITION XXXVIII.

Fig. 13. If a tangent cut any diameter, and if, from the point of contact, an ordinate be drawn to that diameter, then as the diſtance between that ordinate and center is to the abſciſſa, ſo is the diameter leſs by the abſciſſa, to the ſubtangent on the diameter continued; that is, $\mathrm{CP}:\mathrm{PF}::\mathrm{YP}::\mathrm{PT}$.

DEMONSTRATION.

Let GQ be an indefinitely ſmall part of the curve, and continued till it cut the diameter produced in T. Draw the ordinate GP, and parallel to it Q*o*, and G*r*, parallel to the diameter FY. Put $\mathrm{YF}=\mathrm{D}$, $\mathrm{FP}=x$, $\mathrm{GP}=y$, $\mathrm{G}r=n$, $\mathrm{Q}r=m$, and $\mathrm{FT}=a$. Then $\mathrm{YP}=\mathrm{D}-x$, $\mathrm{Y}o=\mathrm{D}-x-n$, $o\mathrm{F}=x+n$, $\mathrm{Q}o=y+m$, and $\mathrm{PT}=a+x$. Then (by ſimilar Δs) $m:n::y:a+x$, $\therefore a+x=n\times\frac{y}{m}$, and (by *Prop.* 36)

$D : P :: Dx - x^2 : y^2$. Alſo $D : P :: Dx - x^2 + Dn - 2xn - n^2 : y^2 + 2ym + m^2$, $\therefore$ $Dy^2 = PDx - Px^2$, and $Dy^2 + 2Dym = PDx - Px^2 + PDn - 2Pxn$, $\therefore$ $PDx - Px^2 + PDn - 2Pxn - 2Dym = (Dy^2 =)\ PDx - Px^2$, and $PDn - 2Pxn = 2Dym$; therefore $n = \frac{2Dym}{PD - 2Px}$. But $a + x = n \times \frac{y}{m}$, $\therefore$ $a + x = \left(\frac{2Dym}{PD - 2Px} \times \frac{y}{m} = \frac{Dy^2}{P} \times \frac{2}{D - 2x} =\right.$ (becauſe by 36. $\frac{Dy^2}{P} = Dx - x^2$) $Dx - x^2 \times \frac{2}{D - 2x} = \frac{2Dx - 2x^2}{D - 2x} =)\ \frac{Dx - x^2}{\frac{1}{2}D - x}$; that is, $\frac{1}{2}D - x : x :: D - x : x + a$, or $CP : PF :: YP :: PT$. *Q. E. D.*

PROPOSITION XXXIX.

IF a tangent interſect any diameter, and, from the point of contact, an ordinate be drawn to that diameter; as the ſemi-diameter leſs by the abſciſſa, is to the ſemi-diameter; ſo is the ſemi-diameter, to the ſemi-diameter added to the external part of the diameter produced to the interſection of the tangent; *i. e.* $CP : CF :: CF : CT$. Fig. 13.

DEMONSTRATION.

$CP + PT = CT$. But $CP = \frac{1}{2}D - x$, and PT (by the laſt) $= \frac{Dx - x^2}{\frac{1}{2}D - x}$; alſo $CT = \frac{1}{2}D + a$, $\therefore$ $\frac{1}{2}D + a = \left(\frac{1}{2}D - x + \frac{Dx - x^2}{\frac{1}{2}D - x} =\right)\ \frac{\frac{1}{4}D^2}{\frac{1}{2}D - x}$, and $\frac{1}{2}D - x : \frac{1}{2}D :: \frac{1}{2}D : \frac{1}{2}D + a$; or, $CP : CF :: CF : CT$. *Q. E. D.*

PROPOSITION XL.

THE ſame things being ſuppoſed as in the laſt, it will be, as the ſemi-diameter leſs by the abſciſſa, is to the ſemi-diameter; ſo is the abſciſſa, to the external part of the diameter produc'd to the interſection of the tangent; *i. e.* $CP : CF :: PF : FT$.

DEMONSTRATION.

By *Prop.* 39. $\frac{\frac{1}{4}D^2}{\frac{1}{2}D - x} = \frac{1}{2}D + a$, $\therefore$ $\frac{1}{4}D^2 = \frac{1}{4}D^2 + \frac{1}{2}Da - \frac{1}{2}Dx - xa$, and $a = \frac{\frac{1}{2}Dx}{\frac{1}{2}D - x}$; that is, $\frac{1}{2}D - x : \frac{1}{2}D :: x : a$; or $CP : CF :: PF :: FT$.

PRO-

PROPOSITION XLI.

AS the ſemi-diameter leſs by the abſciſſa is to the ſemi-diameter, ſo is the diameter leſs by the abſciſſa, to the diameter added to the external part of the diameter produced to the tangent; *i. e.* CP : CF :: YP :: YT.

DEMONSTRATION.

By *Prop.* 40, $a=\frac{\frac{1}{2}Dx}{\frac{1}{2}D-x}$, therefore $D+a=(D+\frac{\frac{1}{2}Dx}{\frac{1}{2}D-x}=)$ $\frac{\frac{1}{2}D^2-\frac{1}{2}Dx}{\frac{1}{2}D-x}$; *i. e.* $\frac{1}{2}D-x : \frac{1}{2}D :: D-x : D+a$, or CP : CF :: YP : YT. *Q. E. D.*

PROPOSITION XLII.

Fig. 13.

AS the diameter leſs by the abſciſſa, is to the diameter added to the external part, ſo is the abſciſſa, to the external part of the diameter produc'd to the tangent; *i. e.* YP : YT :: PF : FT.

DEMONSTRATION.

By *Prop.* 40, $\frac{1}{2}D-x : \frac{1}{2}D :: x : a$; and (by *Prop.* 41.) $\frac{1}{2}D-x : \frac{1}{2}D :: D-x : D+a$, $\therefore$ (by equality) $D-x : D+a :: x : a$; or YP ;: YT :: PF : FT.

PROPOSITION XLIII.

AS the ſemi-diameter added to the external part, is to the ſemi-diameter; ſo is the external part, to the abſciſſa; *i. e.* CT : CF :: FT : FP.

DEMONSTRATION.

By *Prop.* 40, $\frac{1}{2}Da-xa=\frac{1}{2}Dx$, $\therefore x=\frac{\frac{1}{2}Da}{\frac{1}{2}D+a}$; that is, $\frac{1}{2}D+a : \frac{1}{2}D :: a : x$, or CT : CF :: TF : PF.

PROPOSITION XLIV.

AS the ſemi-diameter added to the external part, is to the diameter added to the external part; ſo is the external part, to the ſubtangent; *i. e.* CT : YT :: FT :: PT.

De-

DEMONSTRATION.

By *Prop.* 43, $x=\frac{\frac{1}{2}Da}{\frac{1}{2}D+a}$, therefore $x+a=\left(\frac{\frac{1}{2}Da}{\frac{1}{2}D+a}+a\right)=\frac{Da+a^2}{\frac{1}{2}D+a}$; that is, $\frac{1}{2}D+a : D+a :: a : x+a$, or CT : YT :: FT : PT. *Q. E. D.*

PROPOSITION XLV.

AS the ſemi-diameter added to the external part, is to the ſemi-diameter; ſo is the diameter added to the external part, to the diameter leſs by the abſciſſa; that is, C T : CF :: Y T : Y P.

DEMONSTRATION.

By *Prop.* 41, $\frac{1}{2}D-x : \frac{1}{2}D :: D-x : D+a$; and (by *Prop.* 39.) $\frac{1}{2}D-x : \frac{1}{2}D :: \frac{1}{2}D : \frac{1}{2}D+a$, $\therefore$ $\frac{1}{2}D+a : \frac{1}{2}D :: D+a : D-x$; or CT : CF :: YT : YP. *Q. E. D.*

PROPOSITION XLVI.

Fig. 13.

AS the diameter leſs by the abſciſſa, is to the ſemi-diameter; ſo is the ſubtangent, to the external part of the diameter produced to the tangent; that is, Y P : CF :: P T : F T.

DEMONSTRATION.

By *Prop*· 40, $\frac{1}{2}Da-xa=\frac{1}{2}Dx$, $\therefore$ $\frac{1}{2}D-x=\frac{\frac{1}{2}Dx}{a}$; and $D-x = \left(\frac{1}{2}D+\frac{\frac{1}{2}Dx}{a}=\right)\frac{\frac{1}{2}Da+\frac{1}{2}Dx}{a}$; that is, $D-x : \frac{1}{2}D :: a+x : a$; or YP : CF :: PT : FT. *Q. E. D.*

PROPOSITION XLVII.

AS the diameter added to the external part, is to the ſemi-diameter; ſo is the ſubtangent, to the abſciſſa; *i. e.* Y T : CF :: P T : PF.

DEMONSTRATION.

By *Prop.* 40, $\frac{1}{2}Da=\frac{1}{2}Dx+xa$; therefore $\frac{1}{2}D+a=\frac{\frac{1}{2}Da}{x}$; and $D+a=\left(\frac{1}{2}D+\frac{\frac{1}{2}Da}{x}=\right)\frac{\frac{1}{2}Dx+\frac{1}{2}Da}{x}$; that is, $D+a : \frac{1}{2}D :: x+a : x$, or YT : CF :: PT : PF. *Q. E. D.*

PROPOSITION XLVIII.

IF, from the extremities of two conjugate diameters, ordinates be drawn to the axis; then the diſtance on the axis between the center and one of theſe ordinates, is a mean proportional between the ſegments of the axis made by the other ordinate; that is, AG : CH : : CH : GB; or, AH : CG : : CG : HB.

DEMONSTRATION.

Fig. 14. Draw the tangent FT, which will be parallel to CD by *Prop.* 34. And let BC$=t$, CH$=a$, GT$=s$, and CG$=x$. Then GB $=t-x$, and AG$=t+x$; and (by the 4. and 22. *E.* 6.) GTq : CHq : : (FGq : DHq : : by *Prop.* 1.) AG×GB : AH×HB. But (by 5. *E.* 2.) AG×GB=BCq−CGq, and AH×HB=BCq−CHq; ∴ GTq : CHq : : CBq−CGq : CBq−CHq; that is, $s^2 : a^2 :: t^2-x^2 : t^2-a^2$. But (by *Prop.* 13.) CG:GB::AG:GT; that is, $x : t-x :: t+x : s$; ∴ $s=\frac{t^2-x^2}{x}$, and $s^2=\frac{\overline{t^2-x^2}\times\overline{t^2-x^2}}{x^2}$, conſequently $\frac{\overline{t^2-x^2}\times\overline{t^2-x^2}}{x^2} : t^2-x^2 :: a^2 : t^2-a^2$; and if you divide the two firſt terms by t^2-x^2, $\frac{t^2-x^2}{x^2} : 1 :: a^2 : t^2-a^2$; and, by compoſition, $t^2 : a^2 :: (1+\frac{t^2-x^2}{x^2}=)\frac{t^2}{x^2} : \frac{t^2-x^2}{x^2}$; and if you multiply the two laſt terms by x^2, $t^2 : a^2 :: t^2 : t^2-x^2$, ∴ $a^2=t^2-x^2$, and $t+x : a :: a : t-x$; or, AG : CH : : CH : GB. In like manner we may prove that AH : CG : : CG : HB.

COROLLARY I.

Hence it is eaſy to draw a conjugate diameter, without drawing a tangent. For, if you produce the ordinate FG, to I in the circumference of a circle on the tranſverſe axis, and make CH=GI, then, from H, draw the ordinate HD, and laſtly, from the point D, thro' the center, draw DCX, and it will be the conjugate diameter required.

COROLLARY II.

The ſum of the ſquares of any two diameters, as (XD and FY) is equal to the ſum of the ſquares of the tranſverſe and conjugate axes.

For if a be put for CG, then (by *Prop.* 1.) $t^2 : c^2 :: ($AH × HB =

$=$by this *Prop.* CGq$=$) a^2 : HDq$=\frac{c^2a^2}{t^2}$; and (by this *Prop.*) CHq$=$ (AG×GB$=$) $\frac{1}{4}t^2-a^2$; ∴ (by 47. *E.* 1.) CDq$=\frac{1}{4}t^2-a^2+\frac{c^2a^2}{t^2}$; alſo (by *Prop.* 1.) $t^2 : c^2 : :$ (AG×GB$=$) $\frac{1}{4}t^2-a^2$: FGq $=\frac{1}{4}c^2-\frac{a^2c^2}{t^2}$, ∴ CFq$=a^2+\frac{1}{4}c^2-\frac{a^2c^2}{t^2}$, whence CDq$+CFq=$ $\frac{1}{4}t^2+\frac{1}{4}c^2$.

PROPOSITION XLIX.

IF any ordinate to the axis, as GF, be produced to the periphery of a circle on the tranſverſe axis, as to I, and if, from the points F and I, tangents be prawn to the reſpective curves, they will both interſect the axis produced in one and the ſame point T. Fig. 15.

DEMONSTRATION.

Draw the radius CI, and put BG$=x$, AB$=t$, BT$=a$, and GI $=y$. Then TG×GC$=$ (GIq$=$) GB×GA; *i. e.* $\overline{a+x}\times\overline{\frac{1}{2}t-x}=(y^2=)\ x\times\overline{t-x}$, ∴ $\frac{1}{2}tx=\frac{1}{2}ta-ax$, or $\frac{1}{2}t-x : \frac{1}{2}t : : x : a$. But in (the *Ellipſe*, by *Prop.* 15.) $\frac{1}{2}t-x : \frac{1}{2}t : : x : a$. In both curves the three firſt terms are the ſame, therefore the fourth term, *viz.* a, $=$BT, is the ſame; and conſequently the point T is that wherein both tangents will interſect. *Q. E. D.*

COROLLARY I.

Hence any point in the curve being given, we have an eaſy method of drawing a tangent to touch that point; for if, from the given point F, you draw the ordinate FG, and produce it to the periphery of the circumſcribing circle in the point I, and draw a tangent touching the circle in that point, as IT, then the point T, where that tangent cuts the axis produced, is the point, to which if, from the given point in the Ellipſe (*viz.* F) you draw a right line, it will be a tangent.

COROLLARY II.

Hence alſo if, from the point T, given in the axis produced, it be required to draw a tangent to the ellipſe, 'tis eaſily done. For if, on CT, you deſcribe the ſemi-circle CIT, and obſerve its interſection, I, with the circle deſcribed on the tranſverſe axis; then if BZ be made equal to BI, and IZ be drawn, and if, from the point F, where that line cuts the curve, the ſtreight line TF be drawn, it will touch the curve in the point F.

SCHOLIUM.

From this Propoſition it is evident, that all the properties of tangents which have been demonſtrated in the ellipſe from *Prop.* 13. to *Prop.* 21, incluſively, hold good alſo in the circle.

PROPOSITION L. *Problem.*

Fig. 16. FROM any given point, as T, any where without the ellipſe, to draw a tangent.

CONSTRUCTION.

From the given point T, through the center, draw the right line TFCY; and, to the diameter YF (by *Corol.* to *Prop.* 48.) draw the conjugate diameter DX; then at pleaſure make the angle YTS, and on TS, ſet TR=TC, and SR=CF; join RF, and, parallel to it, draw SP; laſtly, through P, and parallel to the conjugate diameter, draw GN; then if, from the point T, to G or N, right lines be drawn, they will touch the ellipſe in thoſe points.

DEMONSTRATION.

By *Conſtruction*, and 2. *Eu.* 6. TR : RS : : TF : FP; but TR =TC, and RS=CF, ∴ TC : CF : : FT : FP, and (by *Prop.* 43.) TG, or TN are tangents.

PROPOSITION LI.

Fig. 17. IF any ordinate to the axis (as Vx) be continued to a point (N) in the focal tangent (TO), then the diſtance (VN) from the axis to that point in the tangent, ſhall be equal to (Kx) the diſtance from the focus to the extremity of that ordinate.

DEMONSTRATION.

Fig. 17. Put CK$=b$, BC$=c$, CV$=d$; then AK$=b+c$, BK$=c-b$, VK $=b\pm d$, BV$=c\pm d$, and AV$=c\mp d$; and K being the focus, (by *Prop.* 4.) KL will be half the parameter of the axis; and (by *Prop.* 3.) CB : AK : : KB : KL; or $c : c+b : : c-b : \frac{c^2-b^2}{c}=KL=\frac{1}{2}p$. Alſo CK : CB : : CB : CT, or $b : c : : c : \frac{c^2}{b}=$CT, by *Prop.* 14. But CT −CK=KT; that is, $\frac{c^2}{b}-b=\frac{c^2-b^2}{b}=$KT. Alſo CT±CV=VT,

that is, $\frac{c^2}{b} \pm d = \frac{c^2 \pm bd}{b} =$ VT. But (by ſimilar △s) KT : KL : : VT : VN; or $\frac{c^2-b^2}{b} : \frac{c^2-b^2}{c} :: \frac{c^2 \pm bd}{b} : \frac{c^2 \pm bd}{c} =$ VN.

2. By *Prop.* 2. CB : KL : : AV × VB : Vxq; or $c : \frac{c^2-b^2}{c} :: c^2 - d^2 : \frac{c^4-b^2c^2-c^2d^2+d^2b^2}{c^2} =$ Vxq; and VKq$=b^2 \pm 2bd+d^2$. But VKq+Vxq=Kxq; that is, $\frac{c^4 \pm 2bdc^2+d^2b^2}{c^2} =$ Kxq, and (by extracting the ſquare root) $\frac{c^2 \pm bd}{c} =$ K x = (by the firſt part) V N. *Q. E. D.*

COROLLARY.

The conjugate axis continued from the center to the focal tangent, is equal to the ſemi-tranſverſe axis; *i. e.* CZ= (KE=) CB.

PROPOSITION LII.

IF perpendiculars be drawn from the vertices to the focal tangent, then theſe perpendiculars ſhall be equal to the diſtance (in the axis) from each vertex to its adjacent focus reſpectively; that is, AO =AK, and BQ=BK.

DEMONSTRATION.

By *Prop.* 24. AO × BQ=AK × KB, ∴ AO : KA : : BK : BQ; but AO=AK, by *Prop.* 51, ∴ KB=BQ. *Q. E. D.*

PROPOSITION LIII.

IF, from the point of contact of the focal tangent, a right line be drawn to the vertex, and any ordinate be produced to the tangent, and cut that line, then the diſtance between the tangent and interſection of theſe lines, is equal to the diſtance (in the axis) from the focus to the application of the ordinate; *i. e.* DN=KV.

DEMONSTRATION.

The △LDN is ſimilar to △LOA, ∴ OA : DN : : (LO : LN : :) KA : KV; but by *Prop* 51, AO=AK, therefore DN=KV. *Q. E. D.*

PRO-

PROPOSITION LIV.

Fig. 18. IF, from any point (P) of the conjugate axis, a right line, PO, equal to the difference of the ſemi-tranſverſe aud ſemi-conjugate, be applied to the tranſverſe axis, and from thence continued, ſo that the external part, OF, be equal to the ſemi-conjugate axis, then, I ſay, the extremity, F, of that line ſhall be in the curve of the ellipſe.

DEMONSTRATION.

Put $CO = b$, $OG = d$, $CG = (b + d =)\ x$, and the other ſymbols as uſual; then $PO = \tfrac{1}{2}t - \tfrac{1}{2}c$, and $OF = \tfrac{1}{2}c$. Then (from ſimilar $\triangle$s) $b : d :: \tfrac{1}{2}t - \tfrac{1}{2}c : \tfrac{1}{2}c$, and (by compoſition) $(b + d =)\ x : d :: \tfrac{1}{2}t : \tfrac{1}{2}c$, $\therefore x^2 : d^2 :: \tfrac{1}{4}t^2 : \tfrac{1}{4}c^2$, and $\dfrac{\tfrac{1}{4}c^2 \times x^2}{\tfrac{1}{4}t^2} = (d^2 =$ by 47. *E.* 1.) $\tfrac{1}{4}c^2 - y^2$, conſequently $y^2 = \left(\tfrac{1}{4}c^2 - \dfrac{\overline{\tfrac{1}{4}c^2 \times x^2}}{\tfrac{1}{4}t^2} =\right) \dfrac{\overline{\tfrac{1}{4}t^2 - x^2} \times \tfrac{1}{4}c^2}{\tfrac{1}{4}t^2}$; $\therefore \tfrac{1}{4}t^2 : \tfrac{1}{4}c^2 :: \overline{\tfrac{1}{2}t + x} \times \overline{\tfrac{1}{2}t - x} : y$; or $ACq : EDq :: AG \times GB : GFq$. *Q. E. D.*

PROPOSITION LV.

Fig. 19. IF a circle be drawn on the tranſverſe axis of the ellipſe, and ordinates be drawn to both curves; it will be, as the tranſverſe axis is to the conjugate, ſo is any ordinate in the circle, to its correſponding ordinate in the ellipſe; that is, $AB : DE :: sq : sr$.

DEMONSTRATION.

By *Prop.* 1. $ABq : DEq ::$ $As \times sB =$ by 35. *E.* 3.) $\overline{sq}|^2 : \overline{sr}|^2$, $\therefore AB : DE :: sq : sr$. *Q. E. D.*

PROPOSITION LVI.

Fig. 19. AS the tranſverſe axis is to the conjugate axis, ſo is the area of a circle on the tranſverſe axis, to the area of the ellipſe.

DEMONSTRATION.

In the following demonſtrations, let ⊙ t be the circle on the tranſverſe, ⊙ c the circle on the conjugate, ⬭ the ellipſe, and ⊙ $\surd\, tc$ the circle whoſe diameter is the ſquare root of t into c; then,

By *Prop.* 55. $t : c ::$ ($sq : sr ::$ by 12. *E.* 5. all the sqs : all the srs) ⊙ t : ⬭. *Q. E. D.*

PROPOSITION LVII.

THE area of every ellipſe is equal to the area of a circle, whoſe diameter is the ſquare root of the tranſverſe axis into the conjugate.

DEMONSTRATION.

By *Prop.* 56. $\odot t : \bigcirc :: (t : c :: t^2 : ct ::$ by 2. *E.* 12.) $\odot t : \odot \sqrt{tc}, \therefore \bigcirc = \odot \sqrt{tc}$. *Q. E. D.*

COROLLARY.

$\odot t : t^2 :: \bigcirc : tc$; that is, as circles are to the ſquares of their diameters, ſo are ellipſes to the rectangles under their tranſverſe and conjugate axes.

PROPOSITION LVIII.

EVERY ellipſe is a mean proportional between the circle on its tranſverſe and the circle on its conjugate axis.

DEMONSTRATION.

By *Prop.* 56. $\odot t : \bigcirc :: (t : c :: tc : c^2 ::$ by 2 *E.* 12. $\odot \sqrt{tc} : \odot c ::$ by *Prop.* 57.) $\bigcirc : \odot c$. *Q. E. D.*

PROPOSITION LIX.

ELLIPSES are to each other in a ratio compounded of the ſub- Fig. 20.
duplicate ratio of their parameters, and ſeſquiplicate ratio of and 21.
their tranſverſe axes directly.

DEMONSTRATION.

By *Prop.* 57. $E = \odot \sqrt{TC}$, and $e = \odot \sqrt{tc}, \therefore E : e :: (\odot \sqrt{TC} : \odot \sqrt{tc} ::$ by 2. *E.* 12. $TC : tc ::$ becauſe $c = \sqrt{tp}$) $T^{\frac{3}{2}} \times P^{\frac{1}{2}} : t^{\frac{3}{2}} \times p^{\frac{1}{2}}$. *Q. E. D.*

PROPOSITION LX.

PArallelograms drawn with their ſides parallel to the conjugate Fig. 22.
diameters, and circumſcribing the ellipſe, are equal.

DEMONSTRATION.

On the tranſverſe axis deſcribe the circle kND; continue the ordinate through the point of contact to I; draw the ordinate MX, and Cd perpendicular to the tangent; then, I ſay, $CD \times CM = Cx \times Ck$.

$\times Ck$. For, if for $GI=$ (by *Prop.* 48.) CX we put b, $Cn=d$, $CM=D$, $Cd=p$, $GF=y$, $Cx=c$, and $Ck=t$; then,

By *Prop.* 55. $y:b::c:t$, $\therefore y=\frac{bc}{t}$, and (by *Prop* 39) $\frac{bc}{t}$ $(=y)$ $:c::c:d$; alſo by ſimilar $\triangle$s $b:D::p:d$, $\therefore \frac{ct}{b}=(d=)\frac{Dp}{b}$, or $ct=Dp$; *i. e.* $Cd\times CM=Cx\times Ck$. *Q. E. D.*

PROPOSITION LXI.

Fig. 17. AS the diſtance between the foci, is to the tranſverſe axis, ſo is the diſtance between the foeus and the vertex, to the diſtance between the vertex and interſection of the focal tangent with the axis produced; *i. e.* $KH:AB::BK:BT$.

DEMONSTRATION.

By *Prop.* 17. $AK:BK::AT:BT$, $\therefore AK-(BK=)AH:BK::AT-BT:BT$; that is, $HK:BK::AB:BT$. *Q. E. D.*

PROPOSITION LXII.

IF a right line be drawn from the focus to any point of the curve, and, from that point, a line be drawn parallel to the axis, and continued to the perpendicular which cuts the axis produced in the point of interſection of the focal tangent; then theſe two lines are in a conſtant ratio, *viz.* as the diſtance between the foci is to the tranſverſe axis; *i. e.* $KE:En::KH:AB$. Fig. 17.

DEMONSTRATION.

By *Prop.* 51. $Cz=KE$, and, by *Prop.* 52, $BQ=BK$; but (from ſimilar $\triangle$s) $Cz:TC::BQ:BT$; that is, $KE:En::(BK:BT$ $::$ by *Prop.* 61.) $HK:AB$. *Q. E. D.*

PROPOSITION LXIII.

THE focal diſtance of any point in the curve, is to a perpendicular let fall from that focus to the tangent of the ſaid point, as the ſemi-conjugate diameter, to the ſemiconjugate axis.

DEMONSTRATION.

Fig. 23. The $\triangle$s FHI, FKL are ſimilar, $\therefore HF:FK::HI:LK$; *i. e.* $HF+FK:HI+LK::BC:CO::FH:HI::BC\times CD:CO\times CD$; but (by *Prop.* 60.) $OC\times CD=BC\times CE$, $\therefore FH:HI::BCD:BCE:CD:CE$. *Q. E. D.*

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with the radius CK, you interſect that perpendicular in the points D and E, right lines paſſing through the points CE, CD, are called aſymptotes; and the perpendicular intercepted between them (as ED) is called the conjugate axis.

PROPOSITION I.

Fig. 3.

AS the ſquare of the tranſverſe axis, is to the ſquare of the conjugate axis; ſo is the rectangle of the tranſverſe added to the abſciſſa, into the abſciſſa, to the ſquare of the ordinate applied to that abſciſſa; that is, ABq : DEq : : BG × AG : GFq.

DEMONSTRATION.

Put $AC=\frac{1}{2}t$, $AE=\frac{1}{2}c$, $CG=x$, $CK=CH=b$, $GP=y$; then $GK=x \backsim b$, and $KH=2b$, and let $FK=z$; then (by the *Geneſis*) $FH=t+z$, and AEq+ACq= (CEq=) CKq; that is, $\frac{1}{4}c^2+\frac{1}{4}t^2=b^2$ by 47. *E.* 1. and (by 12. and 13. *E.* 2.) HFq=KHq+FKq$\pm\overline{2KH \times GK}$; that is, $t^2+2tz+z^2=z^2+4b^2+4bx-4b^2$, $\therefore z=\frac{4bx-t^2}{2t}$; and by ſquaring both ſides, $\frac{16b^2x^2+t^4-8bt^2x}{4t^2}$ $=(z^2=)\,y^2+x^2-2bx+b^2$; which being clear'd of fractions and contradictory terms, will become $16b^2x^2+t^4=4t^2y^2+4t^2x^2+4t^2b^2$; and if, for $16b^2$ and $4b^2$ in this equation, we ſubſtitute their reſpective values in the firſt, and throw away contradictory terms, and divide by 4, we ſhall have $t^2y^2=c^2x^2-\frac{1}{4}t^2c^2$, which reduced to an analogy, gives $t^2 : c^2 : : \overline{x+\frac{1}{2}t} \times \overline{x-\frac{1}{2}t} : y^2$; or ABq : DEq : : BG × AG : FGq. *Q. E. D.*

COROLLARY.

Let the tranſverſe and conjugate axes be repreſented by t, and c, any abſciſſa, and its ordinate by x, and y; then by this Theorem $t^2 : c^2 : : \overline{t+x} \times x : y^2$, $\therefore t^2y^2=c^2tx+c^2x^2$, which is the equation of the curve.

DEFINITION.

A third proportional to the tranſverſe and conjugate axis, is called the parameter of the axis; that is, if p be put for the parameter, $t : c : : c : p$, $\therefore tp=c^2$.

PROPOSITION II.

AS the tranſverſe axis is to the parameter of the axis, ſo is the tranſverſe added to any abſciſſa, into that abſciſſa, to the ſquare of the ordinate apply'd to that abſciſſa; that is, $t : p :: \overline{t+x} \times x : y^2$.

DEMONSTRATION.

By the conſtruction of the parameter, $tp = c^2$; and if, in the equation of the curve, we ſubſtitute tp, for cc, we ſhall have $ty^2 = tpx + px^2$ (the equation of the curve in the terms of the parameter) which being reduced to an analogy gives $t : p :: \overline{t+x} \times x : y^2$. *Q. E. D.*

COROLLARY.

As the rectangle of the tranſverſe added to any abſciſſa into that abſciſſa, is to the ſquare of the ordinate applied to that abſciſſa; ſo is the rectangle of the tranſverſe added to any other abſciſſa into that abſciſſa, to the ſquare of the ordinate apply'd to that abſciſſa; for (by this *Prop.*) $\overline{t+x} \times x : y^2 :: (t : p ::) \overline{t+X} \times X : Y^2$.

PROPOSITION III.

As half the tranſverſe axis, is to the ſum of the tranſverſe and focal diſtance; ſo is the focal diſtance, to half the parameter of the axis; that is (by putting q for the focal diſtance) $\frac{1}{2}t : \frac{1}{2}t + q :: q : \frac{1}{2}p$.

DEMONSTRATION.

CK $(= \mathrm{CE}) - \mathrm{AC} = \mathrm{AK}$; that is, $\sqrt{\frac{1}{4}t^2 + \frac{1}{4}c^2} - \frac{1}{2}t = q$; but (by *Prop.* 2.) $\frac{1}{4}c^2 = \frac{1}{4}pt$, $\therefore \sqrt{\frac{1}{4}t^2 + \frac{1}{4}tp} - \frac{1}{2}t = q$; and $\frac{1}{4}tp = tq + q^2$; *i.e.* $\frac{1}{2}t : t + q :: q : \frac{1}{2}p$. *Q. E. D.* Fig. 3.

PROPOSITION IV.

The parameter of the axis is equal to double the ordinate paſſing through the focus; that is, (if y be put for the ordinate paſſing through the focus) $y = \frac{1}{2}p$, or $p = 2y$.

DEMONSTRATION.

By *Prop.* 2. (if you put q for the focal diſtance) $t : p :: \overline{t+q} \times q : y^2$; and (by *Prop.* 3.) $\overline{t+q} \times q = \frac{1}{4}tp$, $\therefore$ (by ſubſtitution) $t : p :: \frac{1}{4}tp : (\frac{1}{4}p^2 =) y^2$, and $\frac{1}{2}p = y$. *Q. E. D.*

PROPOSITION V.

As the ſum of the tranſverſe axis and its parameter, is to the diſtance between the foci; ſo is the diſtance between the foci, to the tranſverſe axis.

DEMONSTRATION.

Put KH$=b$; then $\frac{1}{2}b=$ ($\frac{1}{2}$KH$=$CK$=$CE) $\sqrt{\frac{1}{4}t^2+\frac{1}{4}c^2}$; and $\frac{1}{4}b^2=\frac{1}{4}t^2+\frac{1}{4}c^2$, or $b^2=t^2+c^2$. But (by *Prop.* 2.) $tp=c^2$, $\therefore b^2=t^2+tp$; that is, $t+p : b :: b : t$. *Q. E. D.*

PROPOSITION VI.

A Fourth proportional to the conjugate axis, tranſverſe axis, and any ordinate, is a mean proportional between the tranſverſe added to the abſciſſa, and the abſciſſa of that ordinate.

DEMONSTRATION.

Let the fourth proportional be b; then $c : t :: y : b$, and $\frac{ty}{c}=b$. But (by *Prop.* 1.) $t^2 : c^2 :: \overline{t+x} \times x : y^2$; or $t : c :: \sqrt{\overline{t+x} \times x} : y$; therefore $\sqrt{\overline{t+x} \times x}=\left(\frac{ty}{c}=\right) b$. *Q. E. D.*

PROPOSITION VII.

AS the ſquare of any ordinate, is to the rectangle of the tranſverſe added to the abſciſſa into the abſciſſa; ſo is the ſquare of the conjugate axis, to the ſquare of the conjugate axis ſubducted from the ſquare of the diſtance of the foci.

DEMONSTRATION.

Let the diſtance between the foci be b; then $c^2+t^2=b^2$, and $t^2=b^2-c^2$. But (by *Prop.* 1.) $y^2 : \overline{t+x} \times x :: c^2 : (t^2=)\ b^2-c^2$. *Q. E. D.*

PROPOSITION VIII.

AS the ſquare of any ordinate, is to the rectangle of the parameter of the axis into the abſciſſa; ſo is the ſquare of the conjugate axis added to the ſame rectangle, to the ſquare of the conjugate axis; *i. e.* $y^2 : px :: c^2+px : c^2$.

DEMONSTRATION.

By the equation of the curve, $t^2y^2=c^2tx+c^2x^2$; and (by *Prop.* 2.) $\frac{c^2}{p}=t$, $\therefore$ (by ſubſtitution and expunging) $c^2y^2=pxc^2+p^2x^2$; that is, $y^2 : px :: c^2+px : c^2$. *Q. E. D.*

PRO-

PROPOSITION IX.

AS the diſtance from the center to the ordinate drawn from the point of contact of any tangent, is to the abſciſſa of that ordinate; ſo is the ſum of the tranſverſe and abſciſſa, to the ſubtangent; that is, CG : AG :: BG : GT. Fig. 4.

DEMONSTRATION.

Suppoſe Fp, an indefinitely ſmall part of the curve, and produc'd ſo as to cut the axis in T; draw the ordinate FG, and, parallel to it, pq; draw Fr parallel to the axis, and put AT$=a$, F$r=q$, G$=n$, and $rp=m$. Then GT$=a+x$, B$q=t+x+n$, A$q=x+n$, and $pq=y+m$. But $pr:r$F :: FG : GT; that is, $m:n::y:x+a$, $\therefore n\times\frac{y}{m}=x+a$, and (by *Prop.* 2.) $t:p::\overline{t+x+n}\times\overline{x+n}:\overline{y+m}\times\overline{y+m}$; alſo $t:p::\overline{t+x}\times x:y^2$; whence (from the firſt analogy,) $ptx+ptn+px^2+2pxn-2tym=(ty^2=$ from the 2d. analogy$)\ ptx+px^2$, $\therefore ptn+2pxn=2tym$; and $n=\frac{2tym}{pt+2px}$. But $a+x=n\times\frac{y}{m}$, therefore $a+x=\left(\frac{2tym}{pt+2px}\times\frac{y}{m}=\frac{2ty^2}{pt+2px}=\frac{ty^2}{p}\times\frac{2}{t+2x}=tx+x^2\times\frac{2}{t+2x}=\frac{2tx+2x^2}{t+2x}=\right)\frac{tx+x^2}{\frac{1}{2}t+x}$; that is, $\frac{1}{2}t+x:x::t+x:a+x$, or CG : AG :: BG : GT. *Q. E. D.*

PROPOSITION X.

AS half the tranſverſe added to the abſciſſa of the ordinate from the point of contact, is to half the tranſverſe axis; ſo is half the tranſverſe axis, to the diſtance (in the axis produc'd) from the center to the interſection of the tangent; that is, CG : CA :: CA : CT.

DEMONSTRATION.

CT$=$CG$-$GT. But CT$=\frac{1}{2}t-a$, CG$=\frac{1}{2}t+x$; and (by the laſt) GT$=\frac{tx+x^2}{\frac{1}{2}t+x}$, $\therefore \frac{1}{2}t-a=\left(\frac{1}{2}t+x-\frac{tx+x^2}{\frac{1}{2}t+x}=\right)\frac{\frac{1}{4}t^2}{\frac{1}{2}t+x}$, and $\frac{1}{2}t+x:\frac{1}{2}t::\frac{1}{2}t:\frac{1}{2}t-a$, or CG : CA :: CA : CT. *Q. E. D.*

PROPOSITION XI.

AS half the tranſverſe added to the abſciſſa of the ordinate drawn from the point of contact, is to half the tranſverſe axis; ſo is the abſciſſa, to the diſtance from the vertex to the interſection of the tangent; that is, CG : AC :: AG : AT.

DEMONSTRATION.

By the laſt, $\frac{1}{2}t-a=\frac{\frac{1}{4}t^2}{\frac{1}{2}t+x}$, therefore $a=\frac{\frac{1}{2}tx}{\frac{1}{2}t+x}$; and $\frac{1}{2}t+x:\frac{1}{2}t::x:a$; or, CG : AC :: AG : AT. *Q. E. D.*

PROPOSITION XII.

AS half the tranſverſe added to the abſciſſa of the ordinate from the point of contact, is to half the tranſverſe; ſo is the tranſverſe added to the abſciſſa, to the tranſverſe leſs by the external part;
Fig. 6. that is, CG : CA :: BG : BT.

DEMONSTRATION.

By *Prop.* 11. $a=\frac{\frac{1}{2}tx}{\frac{1}{2}t+x}$, $\therefore t-a=(t-\frac{\frac{1}{2}tx}{\frac{1}{2}t+x}=)\frac{\frac{1}{2}t^2+\frac{1}{2}tx}{\frac{1}{2}t+x}$, and $\frac{1}{2}t+x:\frac{1}{2}t::t+x:t-a$; or CG : CA :: BG : BT. *Q. E. D.*

PROPOSITION XIII.

AS the tranſverſe added to the abſciſſa of the ordinate from the point of contact, is to the tranſverſe leſs by the external part; ſo is the abſciſſa to the external part; that is, BG : BT :: GA : AT.

DEMONSTRATION.

By *Prop.* 11. $\frac{1}{2}t+x:\frac{1}{2}t::x:a$; and (by *Prop.* 12.) $\frac{1}{2}t+x:\frac{1}{2}t::t+x:t-a$, $\therefore$ (by equality) $t+x:t-a::x:a$; or BG : BT :: AG : AT. *Q. E. D.*

PROPOSITION XIV.

AS half the tranſverſe leſs by the external part, is to half the tranſverſe; ſo is the external part, to the abſciſſa of the ordinate from the point of contact; that is, CT : CA :: AT : AG.

DEMONSTRATION.

By *Prop.* 11. $a=\frac{\frac{1}{2}tx}{\frac{1}{2}t+x}$; whence $x=\frac{\frac{1}{2}ta}{\frac{1}{2}t-a}$, and $\frac{1}{2}t-a:\frac{1}{2}t::a:x$; or CT : CA :: TA : AG. *Q. E. D.*

PROPOSITION XV.

AS half the tranſverſe leſs by the external part, is to the tranſverſe leſs by the external part; ſo is the external part, to the abſciſſa of the ordinate from the point of contact added to the external part; that is, CT : BT : : AT : GT.

DEMONSTRATION.

By the laſt, $x=\frac{\frac{1}{2}ta}{\frac{1}{2}t-a}$, $\therefore x+a=(a+\frac{\frac{1}{2}ta}{\frac{1}{2}t-a}=)\frac{ta-a^2}{\frac{1}{2}t-a}$; and $\frac{1}{2}t-a:t-a::a:x+a$; or CT : BT : : AT : GT. *Q. E. D.* Fig. 6.

PROPOSITION XVI.

AS the tranſverſe axis added to the abſciſſa of the ordinate from the point of contact, is to half the tranſverſe axis; ſo is the ſame abſciſſa added to the external part, to the external part; *i. e.* BG : CA : : GT : AT.

DEMONSTRATION.

By *Prop.* 11. $\frac{1}{2}t+x=\frac{\frac{1}{2}tx}{a}$; $\therefore t+x=(\frac{1}{2}t+\frac{\frac{1}{2}tx}{a}=)\frac{\frac{1}{2}ta+\frac{1}{2}tx}{a}$, and $t+x:\frac{1}{2}t::x+a:a$; or BC : CA : : TG : AT. *Q. E. D.*

PROPOSITION XVII.

AS the tranſverſe axis leſs by the external part, is to half the tranſverſe axis; ſo is the ſubtangent, to the abſciſſa of the ordinate drawn from the point of contact; that is, BT : CA : : TG : AG.

DEMONSTRATION.

By *Prop.* 14. $\frac{1}{2}t-a=\frac{\frac{1}{2}ta}{x}$; $\therefore t-a=(\frac{1}{2}t+\frac{\frac{1}{2}ta}{x}=)\frac{\frac{1}{2}tx+\frac{1}{2}ta}{x}$, and $t-a:\frac{1}{2}t::x+a:x$; or BT : CA : : TG : AG. *Q. E. D.*

PROPOSITION XVIII.

THE ratio of the ordinate drawn from the point of contact to the ſubtangent, is equal to the ratio compounded of the ratio of the diſtance between the center and ordinate, to the ordinate; and of the ratio of the parameter of the axis, to the axis; that is, $\frac{GF}{GT}=\frac{CG}{FG}\times\frac{p}{t}$.

DEMONSTRATION.

By *Prop.* 9. $tx+x^2=\overline{\frac{1}{2}t+x}\times\overline{x+a}$; and (by *Prop.* 2.) $t:p::(tx+x^2=)\ \overline{\frac{1}{2}t+x}\times\overline{x+a}:y^2$, $\therefore ty^2=p\times\overline{\frac{1}{2}t+x}\times\overline{x+a}$; and (if you divide Fig. 4.

divide by $x+a$) $\frac{ty^2}{x+a}=p\times\frac{1}{2}\overline{t+x}$; and again (if you divide by ty) $\frac{y}{x+a}=\left(\frac{p\times\frac{1}{2}\overline{t+x}}{ty}=\right)\frac{\frac{1}{2}t+x}{y}\times\frac{p}{t}$; or $\frac{GF}{GT}=\frac{CG}{FG}\times\frac{p}{t}$. *Q. E. D.*

PROPOSITION XIX.

IF, from the vertices of the oppoſite ſections, and from the center, perpendiculars be drawn to the axis, and cut any tangent, and alſo an ordinate be drawn from the point of contact, then theſe four lines ſhall be proportional; that is, BO : CP : : FG : AQ.

DEMONSTRATION.

By *Prop.* 15. TB : TC : : TG : AT, ∴ (by 4 *E.* 6.) BO : CP : : FG : AQ. *Q. E. D.*

COROLLARY.

Hence BO×AQ=CP×FG.

PROPOSITION XX.

IF perpendiculars to the axis be drawn from the vertices of the oppoſite ſections, and cut any tangent, the rectangle of theſe perpendiculars ſhall be equal to the rectangle of the greateſt and leaſt diſtance of either focus from the vertex; that is, BO×AQ=KA×KB=AH×BH.

DEMONSTRATION.

Let AQ=m, BO=n, and AK, or BH=q; then (by ſimilar △s,) $m : y$: : ($a : x+a$: : by *Prop.* 16.) $\frac{1}{2}t : t+x$; alſo $n : y$: : ($t-a : x+a$: : by *Prop.* 17.) $\frac{1}{2}t : x$, ∴ $m=\frac{\frac{1}{2}ty}{t+x}$; and $n=\frac{\frac{1}{2}ty}{x}$; and (theſe being multiplyed,) $mn=\frac{\frac{1}{4}t^2y^2}{tx+x^2}$, ∴ $mn : \frac{1}{4}t^2$: : ($y^2 : tx+x^2$: : by *Prop.* 2.) $p : t$; and $mn=$ ($\frac{1}{4}pt=$ by *Prop.* 3.) $\overline{t+q}\times q$; or BO×AQ=AK×KB=AH×BH.

LEMMA.

Fig. 5. If, on the extremities of any chord line AB, perpendiculars, as BQ, AD, be drawn, and if any right-line, as DQ, paſs through the center, and cut theſe perpendiculars, then the external parts OQ, PD, of that line ſhall be equal; and the rectangle of the perpendiculars ſhall be equal to the rectangle of the ſecant QP into the external part QO.

DEMONSTRATION.

Becauſe the angle A is right, $\therefore$ BN is a diameter, and paſſes through the center C, and $\triangle$ CBQ is ſimilar to the $\triangle$ CND, $\therefore$ CB : CN :: CQ : CD :: BQ : ND. But CB=CN, $\therefore$ CQ=CD, BQ=ND, and QP=DO; but (by 36 *E*. 3.) DA×(DN=) BQ=DO×DP=QP×QO. *Q. E. D.*

PROPOSITION XXI.

IF, from the interſections (P, S, of a circle drawn on the tranſverſe, with any tangent, perpendiculars (as P*h*, S*k*) to the tangent be drawn, I ſay they will interſect the tranſverſe axis produc'd in the focal points K and H. Fig. 6.

DEMONSTRATION.

The $\triangle$s, TOB, TP*h*, TAQ, and TS*k* having the angles at T common, and each a right angle, are ſimilar, $\therefore$ BO : P*h* : : S*k* : AQ, and BO×AQ= (P*h*×S*k*= by the preceding *Lemma*) *h*A×*h*B, or *k*B×*k*A. But (by *Prop*. 20.) BO×AQ=HA×HB, or KA×KB, $\therefore$ the points K, *k*, and H, *h* are coincident. *Q. E. D.*

COROLLARY.

KS×PH=$\frac{1}{4}$*pt*, becauſe HA×HB=$\frac{1}{4}$*pt*, by *Prop*. 3.

PROPOSITION XXII.

IF, from any part of the curve, lines be drawn to the foci, and the angle formed by thoſe lines be biſected, then the biſecting line will be a tangent to the curve in the angular point. Fig. 7.

DEMONSTRATION.

Take FX=FK; then (becauſe by hypotheſis $\angle$HFT=$\angle$TFK) if you take any point, S, in the line FT, the line KS=SX, by 4 *E*. 1. Draw SH; then AB (=HX)+(SX+) SK is greater than SH, $\therefore$ the point S is without the curve; for if it were in the curve, AB+SK would be equal to SH, by the Geneſis.

COROLLARY.

Hence lines drawn from the foci, to the point of contact, make with the tangent equal angles.

PROPOSITION XXIII.

A Right line perpendicular to any tangent at the point of contact, biſects the angle made by lines drawn through the point

of

of contact, and from the focus points; that is, if FY be perpendicular to FT, then the $\angle ZFY=\angle HFY$.

DEMONSTRATION.

The $\angle LFY=\angle TFY$ by hypotheſis, and the $\angle LFZ=$(KFT = by *Prop.* 22.) TFH; which being taken from the former, there remains $\angle ZFY=\angle HFY$. *Q. E. D.*

PROPOSITION XXIV.

IF, on the tangent at the point of contact, a perpendicular be drawn, and, from the point where that perpendicular cuts the axis, two lines be drawn perpendicular to the lines which connect the foci to the point of contact, then the diſtance on theſe lines between the point of contact and the perpendiculars, will be equal to half the parameter of the axis; *i. e.* $Fq=Fr=\frac{1}{2}p$.

DEMONSTRATION.

Fig. 9. From the points S, P, where a circle on the tranſverſe cuts the tangent, draw lines to the foci H and K, which (by *Prop.* 21.) will be perpendicular to the tangent, ∴ PK, HX, and YF are parallel: continue HS to X; then (by *Prop.* 22.) HS=SX, and HF=FX, ∴ $KX=AB=t$; alſo $\triangle KFY$ is ſimilar to $\triangle KXH$; and becauſe the $\angle FPK=\angle YqF$, and the angles PKF, and qFY are the complements of the $\angle qFL$, ∴ $\triangle$s PKF and YFq are ſimilar, and $KX : XH :: (KF : FY ::) KP : Fq$, and $KX\times Fq=XH\times KP$, or $\frac{1}{2}KX\times Fq=(\frac{1}{2}XH=)\ SH\times KP$; that is, $\frac{1}{2}t\times Fq=(SH\times KP=$ by *Prop.* 21.) $\frac{1}{4}pt$, or $Fq=\frac{1}{2}p$; but (by 26 *E.* 1.) $Fr=Fq$, ∴ $Fq=Fr=\frac{1}{2}p$. *Q. E. D.*

PROPOSITION XXV.

IF perpendiculars from the vertices cut any tangent, the part of the tangent intercepted between the interſections ſhall be the diameter of a circle whoſe periphery ſhall paſs through the foci.

DEMONSTRATION.

Fig. 10. By *Prop.* 20. $BO\times AQ=HA\times HB$, ∴ $BO : BH :: AH : AQ$. But $\angle QAH=\angle OBH$, ∴ (by 6 *E.* 6.) $\triangle AQH$ is ſimilar to $\triangle OBH$, and $\angle BOH=\angle AHQ$. Alſo $\angle AQH=\angle BHO$; but $\angle AQH+\angle AHQ=$ a right angle; ∴ $\angle QHO =(AHQ+BHO=)$ a right angle, and (by 31 *E.* 3.) OQ is a diameter. In like manner QKO may be proved a right angle. *Q. E. D.*

COR-

COROLLARY.

If OQ be biſected in N, then NO$=$HN$=$NQ.

PROPOSITION XXVI.

IF, from the remoter focus, a right line be drawn to the point of contact, and in that line HX$=$AB, and from the other focus KX be drawn, and cut the tangent in S, then a right line drawn from the center to that interſection will be equal to half the tranſverſe axis; that is, CS$=(\frac{1}{2}$AB$=)$ CA.

DEMONSTRATION.

In the $\triangle$s KCS, KHX, the $\angle$K is common, KC$=$CH, Fig. 11. and (by *Prop.* 22.) KS$=$SX, $\therefore$ (by 6 *E.* 6.) the $\triangle$s are ſimilar, and CS is $\parallel$ to HX; alſo CS$=(\frac{1}{2}$HX$=\frac{1}{2}$AB$)=$CB$=$CA. *Q.E.D.*

PROPOSITION XXVII.

IF, from the remoter focus, a line be drawn to the point of contact, and another from the center $\parallel$ to the tangent; the diſtance between the point of contact, and interſections of theſe two lines, is equal to half the tranſverſe axis; that is, FZ$=\frac{1}{2}$AB.

DEMONSTRATION.

Draw CS $\parallel$ HF; then is the figure FZCS a ▯, $\therefore$ ZF$=($CS $=$by *Prop.* 26.) AC$=\frac{1}{2}$AB. *Q. E. D.*

PROPOSITION XXVIII.

IF, within the curve, lines be drawn parallel to any tangent, they will be biſected by a diameter produc'd through the point of contact. Alſo $\triangle BCS = \triangle CDr - \triangle pdz = \triangle CVo - \triangle Vpx$. Fig. 12.

DEMONSTRATION.

Put $dz=y$, $dp=c$, $Cd=n$, $BS=r$, $dr=p$, $Vx=Y$, $CV=g$, Fig. 12. $Vo=q$, $Vp=b$, and the abſciſſas Bd, $BV=x$, X. Then,

1. By ſimilar $\triangle$s $\frac{y}{c}=\left(\frac{FG}{GT}=\right.$ by *Prop.* 18.) $\frac{CG}{FG}\times\frac{p}{t}$. But $\frac{CG}{GF}=$ $\frac{\frac{1}{2}t}{r}$ by ſimilar $\triangle$s, $\therefore \frac{y}{c}=\frac{\frac{1}{2}t}{r}\times\frac{p}{t}$. Divide by $\frac{p}{t}$, and $\frac{\frac{1}{2}t}{r}=\frac{y}{c}\times\frac{t}{p}$; multiply by cry, and $\frac{1}{2}tcy=ry^2\times\frac{t}{p}$. But (by *Prop.* 2.) $y^2\times\frac{t}{p}=tx+x^2$, $\therefore \frac{1}{2}tcy=r\times\overline{tx+x^2}$, and $rn^2-\frac{1}{2}tcy=(rn^2-r\times\overline{tx+x^2}=)\ r\times\overline{n^2-tx+x^2}$; and by (6 *E.* 2.) $n^2-\overline{tx+x^2}=\frac{1}{4}t^2$, $\therefore rn^2-\frac{1}{2}tcy=r\times\frac{1}{4}t^2$;

$\frac{1}{4}t^2$; divide by $\frac{1}{2}t$, and $\frac{rn^2}{\frac{1}{2}t} - cy = r\times\frac{1}{2}t$. By ſimilar Δs $\frac{1}{2}t : r :: n : p = \frac{rn}{\frac{1}{2}t}$, $\therefore$ (by ſubſtitution) $np - cy = r\times\frac{1}{2}t$, or $\mathrm{C}d\times dr - dp\times dz = \mathrm{BS}\times\mathrm{BC}$; that is, $\Delta\,\mathrm{C}dr - pdz = \Delta\,\mathrm{BCS}$.

2. By ſimilar Δs $\frac{\mathrm{Y}}{b} = \left(\frac{\mathrm{FG}}{\mathrm{GT}} = \text{by } \textit{Prop. } 18.\right) \frac{\mathrm{CG}}{\mathrm{FG}}\times\frac{p}{t}$. By ſimilar Δs $\frac{\mathrm{CG}}{\mathrm{GF}} = \frac{g}{q}$, $\therefore \frac{\mathrm{Y}}{b} = \frac{g}{q}\times\frac{p}{t}$. Divide by $\frac{p}{t}$, and $\frac{g}{q} = \frac{\mathrm{Y}}{b}\times\frac{t}{p}$; multiply by $bq\mathrm{Y}$, and $gb\mathrm{Y} = q\mathrm{Y}^2\times\frac{t}{p}$; but (by *Prop.* 2.) $\mathrm{Y}^2\times\frac{t}{p} = t\mathrm{X}+\mathrm{X}^2$; therefore $gb\mathrm{Y} = q\times\overline{t\mathrm{X}+\mathrm{X}^2}$, and $qg^2 - gb\mathrm{Y} = (qg^2 - q\times\overline{t\mathrm{X}+\mathrm{X}^2} =)\ q\times\overline{g^2 - \overline{t\mathrm{X}+\mathrm{X}^2}}$; but (by 6 *E.* 2.) $g^2 - \overline{t\mathrm{X}+\mathrm{X}^2} = \frac{1}{4}t^2$; $\therefore qg^2 - gb\mathrm{Y} = q\times\frac{1}{4}t^2$. Divide by g, and $qg - b\mathrm{Y} = \frac{q\times\frac{1}{4}t^2}{g}$; but (by ſimilar Δs) $g : q :: \frac{1}{2}t : r = \frac{q\times\frac{1}{2}t}{g}$, $\therefore$ (by ſubſtitution) $qg - b\mathrm{Y} = r\times\frac{1}{2}t$, or $\mathrm{V}o\times\mathrm{CV} - \mathrm{V}p\times\mathrm{V}x = \mathrm{BC}\times\mathrm{BS}$; that is, $\Delta\,\mathrm{CV}o - \Delta\,\mathrm{V}px = \Delta\,\mathrm{BCS} =$ (by the former part) $\Delta\,\mathrm{C}dr - \Delta\,pdz$, and (by tranſpoſition) $\Delta\,\mathrm{CV}o - \Delta\,\mathrm{C}dr = \Delta\,\mathrm{V}px - \Delta\,pdz$.

3. From both ſides of the laſt equation take the figure $dzbo\mathrm{V}$, and there remains the $\Delta\,oxb =$ and ſimilar $\Delta\,bzr$, $\therefore xb = bz$. *Q.E.D.*

PROPOSITION XXIX.

THE $\Delta\,\mathrm{BSC} = \Delta\,\mathrm{CFT}$; alſo the *trapezium* $d\mathrm{BS}r = \Delta\,pdz$, and $\Delta\,bzr =$ *trapezium* $b\mathrm{FT}p$, and $\overline{\mathrm{FT}+bp}\times b\mathrm{F} = zb\times rb$.

DEMONSTRATION.

From ſimilar Δs $\mathrm{BS} : \mathrm{FG} :: (\mathrm{BC} : \mathrm{GC} ::$ by *Prop.* 10.) $\mathrm{CT} : \mathrm{BC}$, $\therefore \mathrm{BS}\times\mathrm{BC} = \mathrm{FG}\times\mathrm{CT}$; or $\Delta\,\mathrm{BSC} = \Delta\,\mathrm{CFT} =$ (by *Prop.* 28.) $\Delta\,\mathrm{C}dr - \Delta\,pdz = \Delta\,\mathrm{CV}o - \Delta\,\mathrm{V}px$, $\therefore$ (by tranſpoſition) $\Delta\,\mathrm{BCS} + \Delta\,pdz = \Delta\,\mathrm{C}dr$; from each ſide take $\Delta\,\mathrm{BCS}$, then there remains $\Delta\,pdz =$ *trapezium* $d\mathrm{BS}r$; and (by tranſpoſing the 1ſt equations) $\Delta\,\mathrm{CFT} + \Delta\,\mathrm{V}px = \Delta\,\mathrm{CV}o$; from each ſide take $\Delta\,\mathrm{CFT} +$ *trapezium* $pbo\mathrm{V}$, and there remains $(\Delta\,obx =)\ \Delta\,bzr =$ *trapezium* $b\mathrm{FT}p$, and (by *Lemma* to *Prop.* 11. of the *Parabola*) $\overline{\mathrm{FT}+bp}\times\mathrm{F}b = zb\times br$. *Q. E. D.*

DE-

DEFINITION.

Let FS : FQ : : $br : bz$: : 2FT : P, the parameter of the diameter FY; then $P = \frac{bz \times 2FT}{br}$; and;

PROPOSITION XXX.

As any diameter is to its parameter (ſo obtained) ſo is the rectangle of the diameter added to the abſciſſa into the abſciſſa, to the ſquare of the ordinate of that abſciſſa; that is, if you put $D = FY$, $x = Fb$, and $y = bz$, or bx, it will be $D : P : : \overline{D+x} \times x : y^2$. Fig. 12.

DEMONSTRATION.

By the *Definition* $P = \frac{y \times 2FT}{br}$, therefore $P \times \frac{\overline{D+x} \times x}{D} = \frac{y \times 2FT}{br} \times \frac{\overline{D+x} \times x}{D} = \frac{y}{br} \times \overline{D+x} \times x \times \frac{2FT}{D}$. But $\frac{2FT}{D} = \left(\frac{FT}{\frac{1}{2}D} =\right)$ by ſimilar Δs $\frac{Tn}{(np=)\,x}$; therefore (by ſubſtitution) $\frac{P \times \overline{D+x} \times x}{D} = \left(\frac{y}{br} \times \overline{D+x} \times Tn = \right.$ (by *Lemma* to *Prop.* 36 of the *Ellipſe)* $\left.\frac{y}{br} \times y \times br = \right) y^2$, $\therefore D : P : : \overline{D+x} \times x : y^2$. *Q. E. D.*

PROPOSITION XXXI.

If a tangent cut any diameter, and if, from the point of contact, an ordinate be drawn to that diameter; then, as the ſemi-diameter added to the abſciſſa, is to the abſciſſa, ſo is the diameter added to the abſciſſa, to the ſubtangent on that diameter; that is, CP : FP : : YP : PT.

DEMONSTRATION.

Let QG be an indefinitely ſmall part of the curve, and produced Fig. 13. to cut the diameter in T; draw the ordinate GP, and, parallel to it, QO; draw Gr parallel to (YO) the diameter continued; and put G$r = n$, Q$r = m$, and FT$= a$. Then YP$= D + x$, YO$= D + x + n$, OF$= x + n$, QO$= y + m$, and PT$+ x + a$; and (by ſimilar Δs) $m : n : : y : x + a$, $\therefore x + a = n \times \frac{y}{m}$. But (by *Prop.* 30.) $D : P : : \overline{D+x+n} \times \overline{x+n} : \overline{y+m} \times \overline{y+m}$; and $D : P : : \overline{D+x} \times x : y^2$; and reducing the firſt analogy into an equation, we ſhall have $PDx +$

$PDn+Px^2+2Pxn-2Dym=$ ($Dy^2=$ in the 2d *Analogy*) $PDx+Px^2$, $\therefore PDn+2Pxn=2Dym$, and $n=\frac{2Dym}{PD+2Px}$. But $x+a = (n \times \frac{y}{m}$, $\therefore x+a = \left(\frac{2Dyu}{PD+2Px} \times \frac{y}{m} = \frac{2Dy^2}{PD+2Px} = \frac{Dy^2}{P} \times \frac{2}{D+2x} = Dx+x^2 \times \frac{2}{D+2x} = \frac{2Dx+2x^2}{D+2x} = \right) \frac{Dx+x^2}{\frac{1}{2}D+x}$, $\therefore \frac{1}{2}D+x : x :: D+x : x+a$; or CP : FP : : YP : PT. *Q. E. D.*

PROPOSITION XXXII.

THE ſame things being ſuppoſed as before, as the ſemi-diameter added to the abſciſſa, is to the ſemi-diameter, ſo is the ſemi-diameter, to the ſemi-diameter leſs by the external part; *i. e.* CP : CF : : CF : CT.

DEMONSTRATION.

$CP-PT=CT$. But $CP=\frac{1}{2}D+x$, $CT=\frac{1}{2}D-a$; and (by *Prop.* 31.) $PT=\frac{Dx+x^2}{\frac{1}{2}D+x}$, therefore $\frac{1}{2}D+x-\frac{Dx+x^2}{\frac{1}{2}D+x}$, or $\frac{\frac{1}{4}D^2}{\frac{1}{2}D+x}=\frac{1}{2}D-a$; that is, $\frac{1}{2}D+x : \frac{1}{2}D :: \frac{1}{2}D : \frac{1}{2}D-a$; or CP : CF : : CF : CT. *Q. E. D.*

PROPOSITION XXXIII.

AS the ſemi-diameter added to the abſciſſa, is to the ſemi-diameter, ſo is the abſciſſa, to the external part; that is, CP : CF : : PF : FT.

DEMONSTRATION.

Fig. 13. By *Prop.* 32. $\frac{\frac{1}{4}D^2}{\frac{1}{2}D+x}=\frac{1}{2}D-a$, $\therefore \frac{1}{4}D^2=\frac{1}{4}D^2+\frac{1}{2}Dx-\frac{1}{2}Da-xa$, and $\frac{1}{2}Da+xa=\frac{1}{2}Dx$, $\therefore \frac{1}{2}D+x : \frac{1}{2}D :: x : a$; or, CP : CF : : PF : FT. *Q. E. D.*

PROPOSITION XXXIV.

AS the ſemi-diameter added to the abſciſſa, is to the ſemi-diameter, ſo is the diameter added to the abſciſſa, to the diameter leſs by the external part; that is, CP : CF : : YP : YT.

DEMONSTRATION.

By *Prop* 33. $a=\frac{\frac{1}{2}Dx}{\frac{1}{2}D+x}$, therefore $D-a= \left(D-\frac{\frac{1}{2}Dx}{\frac{1}{2}D+x}=\right)$

$\frac{1}{2}D^2$

$\frac{\frac{1}{2}D^2+\frac{1}{2}Dx}{\frac{1}{2}D+x}$, and $\frac{1}{2}D+x : \frac{1}{2}D :: D+x : D-a$; or CP : CF :: YP : YT. *Q. E. D.*

PROPOSITION XXXV.

AS the diameter added to the abſciſſa, is to the diameter leſs by the external part, ſo is the abſciſſa, to the external part; *i. e.* YP : YT :: PF : FT.

DEMONSTRATION.

By *Prop.* 33. $\frac{1}{2}D+x : \frac{1}{2}D :: x : a$; and (by *Prop.* 34.) $\frac{1}{2}D+x : \frac{1}{2}D :: D+x : D-a$, ∴ (by equality) $D+x : D-a :: x : a$; or YP : YT :: PF : FT. *Q. E. D.*

PROPOSITION XXXVI.

AS the ſemi-diameter leſs by the external part, is to the ſemi-diameter, ſo is the external part, to the abſciſſa; *i. e.* CT : CF :: FT : FP.

DEMONSTRATION.

By *Prop.* 32. $\frac{\frac{1}{4}D^2}{\frac{1}{2}D+x}=\frac{1}{2}D-a$, $\therefore \frac{1}{4}D^2=\frac{1}{4}D^2-\frac{1}{2}Da+\frac{1}{2}Dx-xa$; and $\frac{1}{2}Dx-xa=\frac{1}{2}Da$; *i. e.* $\frac{1}{2}D-a : \frac{1}{2}D :: a : x$; or, CT : CF :: FT : FP. *Q. E. D.*

PROPOSITION XXXVII.

AS the ſemi-diameter leſs by the external part, is to the diameter leſs by the external part, ſo is the external part, to the ſub-tangent; *i. e.* CT : YT :: FT : PT.

DEMONSTRATION.

Fig. 13.

By *Prop.* 36. $\frac{1}{2}Dx-xa=\frac{1}{2}Da$, $\therefore x=\frac{\frac{1}{2}Da}{\frac{1}{2}D-a}$; and $a+x= \left(a+\frac{\frac{1}{2}Da}{\frac{1}{2}D-a}=\right)\frac{Da-aa}{\frac{1}{2}D-a}$, and $\frac{1}{2}D-a : D-a :: a : x+a$; or CT : YT :: FT : PT. *Q. E. D.*

PROPOSITION XXXVIII.

AS the ſemi-diameter leſs by the external part, is to the ſemi-diameter, ſo is the diameter leſs by the external part, to the diameter added to the abſciſſa; *i. e.* CT : CF :: YT : YP.

DEMONSTRATION.

By *Prop.* 37. $x = \frac{\frac{1}{2}Da}{\frac{1}{2}D - a}$, $\therefore D + x = (D + \frac{\frac{1}{2}Da}{\frac{1}{2}D - a} =) \frac{\frac{1}{2}D^2 - \frac{1}{2}Da}{\frac{1}{2}D - a}$; and $\frac{1}{2}D - a : \frac{1}{2}D :: D - a : D + x$; or CT : CF :: YT : YP. *Q. E. D.*

PROPOSITION XXXIX.

IF any ordinate to the axis (as Vx) be continued to (N, in) the focal tangent (TO), then the diſtance (VN) from the axis to that point in the tangent, ſhall be equal to (Kx) the diſtance from the focus to the extremity of that ordinate.

DEMONSTRATION.

Fig. 14. Put $CK = b$, $CB = c$, $CV = d$, then $AK = b + c$, $BK = b - c$, $VK = d \backsim b$, $BV = d - c$, and $AV = d + c$. Then,

1. The point K being the focus by *Prop.* 4. KL = half the parameter of the axis, and (by *Prop.* 3.) CB : AK :: KB : KL; or $c : b + c :: b - c : \frac{b^2 - c^2}{c} = (KL =) \frac{1}{2}p$. Alſo (by *Prop.* 10.) CK : CB :: CB : CT; or, $b : c :: c : \frac{c^2}{b} = CT$. But CK − CT = KT; *i. e.* $b - \frac{c^2}{b} = \frac{b^2 - c^2}{b} = KT$, and CV − CT = VT, or $d - \frac{c^2}{b} = \frac{db - c^2}{b} = VT$; and (by ſimilar △s) KT : KL :: VT : VN; or $\frac{b^2 - c^2}{b} : \frac{b^2 - c^2}{c} :: \frac{bd - c^2}{b} : \frac{bd - c^2}{c} = VN$.

2. By *Prop.* 2. CB : KL :: AV × VB : Vxq; or $c : \frac{b^2 - c^2}{c} :: d^2 - c^2 : \frac{b^2 d^2 - b^2 c^2 - c^2 d^2 + c^4}{c^2} = Vx\text{q}$; and $VK\text{q} = d^2 - 2db + b^2$. But (by 47. *E.* 1.) $VK\text{q} + Vx\text{q} = Kx\text{q}$; or $\frac{c^4 - 2c^2 bd + b^2 d^2}{c^2} = Kx\text{q}$; and by extracting the ſquare root, $Kx = (\frac{bd - c^2}{c} =$ by the firſt part) VN. *Q. E. D.*

PROPOSITION XL.

Fig. 14. IF perpendiculars be drawn from the vertices to the focal tangent, then theſe perpendiculars ſhall be equal to the diſtance (in the axis)

axis) from each vertex to its adjacent focus reſpectively; that is, AO =AK, and BQ=BK.

DEMONSTRATION.

By *Prop.* 20. AO × BQ=AK × KB, $\therefore$ AO : AK :: KB : BQ. But (by *Prop.* 39.) BQ=BK, $\therefore$ AO=AK. *Q. E. D.*

PROPOSITION XLI.

IF, through the point of contact of the focal tangent, a right line be drawn to the vertex, and any ordinate be produced to the tangent, and cut that line, then the diſtance between the tangent and interſection of theſe lines, is equal to the diſtance (in the axis) from the focus to the application of the ordinate; that is, DN=KV. Fig. 14.

DEMONSTRATION.

From ſimilar $\triangle$s AO : DN :: [(LO : LN :: AL : LD ::) AK : KV; But AO=AK by *Prop.* 40, $\therefore$ DN=KV. *Q. E. D.*

Of the HYPERBOLIC ASYMPTOTES.

PROPOSITION XLII.

IF any ordinate to the axis be continued both ways to the Aſymptotes (as NGP) then the ſquare of the ſemi-conjugate axis (BE) will be equal to the rectangle of the greateſt and leaſt diſtance of either extremity of that line from the curve; that is, BEq=NS × SP=Pr × rN. Fig. 15.

DEMONSTRATION.

Let NG=PG=b, and the other ſymbols as uſual; then CG= $\frac{1}{2}t+x$, and (by ſimilar $\triangle$s) CBq : BEq :: CGq : GNq; *i. e.* $\frac{1}{4}t^2$: $(\frac{1}{4}c^2=)\ \frac{1}{4}tp :: \frac{1}{4}t^2+tx+x^2 : b^2$, $\therefore b^2=\frac{p}{t}\times\overline{\frac{1}{4}t^2+tx+x^2}$. But (by *Prop.* 2) $t : p :: tx+x^2 : y^2$, $\therefore y^2=\frac{p}{t}\times\overline{tx+x^2}$, and $b^2-y^2=\left(\frac{p}{t}\times\frac{1}{4}t^2=\frac{1}{4}pt=\right)\frac{1}{4}c^2$. Alſo $b+y : \frac{1}{2}c :: \frac{1}{2}c : b-y$; or NS : EB :: EB : SP. *Q. E. D.*

PROPOSITION XLIII.

THE Aſymptotes continually approach to the curve.

DEMONSTRATION.

By *Prop.* 42. $EB^2 = NS \times SP = Oa \times aY$, therefore $NS : Oa :: aY : SP$. But NS is leſs than Oa, therefore aY is leſs than SP, and conſequently the point Y is nearer to the curve, than the point P. *Q. E. D.*

PROPOSITION XLIV.

IF the Aſymptotes and curve be infinitely produced, they will never concur.

DEMONSTRATION.

From the two firſt analogies of *Prop.* 42. ii follows, that $\overline{\tfrac{1}{2}t+x}|^2 : b^2 :: \overline{t+x} \times x : y^2$; that is, $CGq : GNq :: AG \times BG : Grq$. But (by 6. *Eu.* 2.) CGq is greater than $AG \times BG$, ∴ GNq is greater than Grq, and GN greater than Gr; and conſequently wherever the point N is taken, it will never touch the curve.

PROPOSITION XLV.

IF an ordinate to the axis be produced both ways to the Aſymptotes, then the parts intercepted on each ſide between the curve and aſymptotes are equal; *i. e.* $SP = rN$.

DEMONSTRATION.

From ſimilar △s, $BD : BE :: GP : GN$; but $BD = BE$, ∴ $GP = GN$, and the ordinates GS, Gr, being equal, rN will be $= SP$.

DEFINITION.

Fig. 16. If the tangent to the vertex of any diameter be continued both ways from the point of contact, with this condition, that as the diameter paſſing through the point of contact, is to its parameter, ſo is the ſquare of the ſemi-diameter, to a fourth proportional; then if the ſquare root of that fourth proportional be ſet both ways from the vertex on the tangent (as FP, FQ) the extremities will determine the conjugate diameter, and if, through theſe extremities, right lines be drawn from the center (as CP, CQ) they ſhall be Aſymptotes.

PROPOSITION XLVI.

Fig. 16. IF any ordinate to a diameter be produced both ways to the Aſymptotes (as mbn) then the ſquare of the ſemi-conjugate diameter will be equal to the rectangle of the greateſt and leaſt diſtance of either extremity of that line from the curve; that is, $FPq = mz \times zn = nr \times rm$.

DEMONSTRATION.

Put $bm=r$, $br=y$, $FP=FQ=\frac{1}{2}c$; then (by the *Definition*) $D:P::CFq:FPq$; or $D:P::\frac{1}{4}D^2:\frac{1}{4}c^2$, $\therefore \frac{1}{4}c^2=\frac{\frac{1}{4}PD^2}{D}=\frac{1}{4}PD$. But (from ſimilar $\triangle$s) $CFq:FPq:Cbq:bmq$; that is, $\frac{1}{4}D^2:\frac{1}{4}DP::\overline{\frac{1}{2}D+x}|^2:r^2$, $\therefore r^2=\frac{P}{D}\times\overline{\frac{1}{4}D^2+Dx+x^2}$; and (by *Prop*. 30.) $D:P::\overline{D+x}\times x:y^2$, $\therefore y^2=\frac{P}{D}\times\overline{Dx+x^2}$, and $r^2-y^2=\left(\frac{P}{D}\times\frac{1}{4}D^2=\right)\frac{1}{4}c^2$. Alſo $r+y:\frac{1}{2}c::\frac{1}{2}c:r-y$; or $FPq=mz\times zn=rn\times rm$.

PROPOSITION XLVII.

THE Aſymptotes drawn through the extremities of any conjugate diameter, and produced, do continually approach the curve.

DEMONSTRATION.

By *Prop*. 46. $mz\times mr=(FPq=)\ wt\times ws$, $\therefore mz:wt::ws:mr$. But mz is leſs than wt, $\therefore ws$ is leſs than mr, and conſequently the point w is nearer the curve than the point m. *Q. E. D.*

PROPOSITION XLVIII.

THE Aſymqtotes, produced through the extremities of the conjugate diameter, will never meet the curve.

DEMONSTRATION.

By *Prop*. 46. $\frac{1}{4}PD=\frac{1}{4}c^2=FPq$; and (by ſimilar $\triangle$s) $\overline{bw}|^2:\overline{Cb}|^2::(FPq:CFq::\frac{1}{4}PD:\frac{1}{4}D^2::)\ P:D$; and (by *Prop*. 30.) $\overline{bs}|^2:Yb\times Fb::PD$; therefore (by equality) $\overline{bs}|^2:Yb\times bF::\overline{bw}|^2:\overline{Cb}|^2$. But by (6. *Eu*. 2.) $\overline{Cb}|^2$ is greater than $Yb\times Fb$, $\therefore \overline{bw}|^2$ is greater than $\overline{bs}|^2$, and bw greater than bs; conſequently wherever the point w be taken in CP produced, it will be without the curve. *Q. E. D.* Fig. 16.

PROPOSITION XLIX.

IF ordinates to any diameter be produced both ways to the aſymptotes, then the external parts between the aſymptotes and the curve are equal; *i. e.* $rm=zn$.

DEMONSTRATION.

From ſimilar $\triangle$s, $PF:FQ::bm:bn$. But $PF=FQ$, $\therefore bm=$

bn; from which if you take away the equal ordinates, there will remain $rm = zn$. *Q. E. D.*

PROPOSITION L.

Fig. 17. IF the right line rt be drawn parallel to the diameter FY, then the ſquare of the ſemi-diameter CF ſhall be equal to the rectangle contained under the greateſt and leaſt diſtance of either extremity of that line from its adjacent aſymptote; that is, $CFq = rd \times rp = pt \times td$.

DEMONSTRATION.

Put $FQ = PF = c$, $FC = t$, $mr = b$, $rp = d$, $rn = p$, and $rd = q$. Then, becauſe $\triangle\, mrp$ is ſimilar $\triangle$ PFC, and $\triangle\, nrd$ is ſimilar $\triangle$ QFC, by 4. *E.* 6. $\frac{b}{d} = \frac{c}{t}$, and $\frac{p}{q} = \frac{c}{t}$, $\therefore \frac{pb}{qd} = \frac{c^2}{t^2}$, or $pb : c^2 :: qd : t^2$. But (by *Prop.* 46.) $bp = c^2$, $\therefore qd = t^2$, or $CFq = rd \times rp$. *Q. E. D.*

PROPOSITION LI.

IF a right line be drawn parallel to any diameter, and cut the oppoſite hyperbolas, then the parts of that line intercepted between the curves and aſymptotes are equal; *i. e.* $rp = td$.

DEMONSTRATION.

Make the abſciſſa $Yo = Fb$; draw the ordinate ot, and the conjugate YR. Then, from ſimilar $\triangle$s, $mr : rp ::$ ($PF = FQ : FC ::$ $YR : YC ::$) $St : td$. But $mr =$ ($zn =$) St, $\therefore rp = td$. *Q. E. D.*

PROPOSITION LII.

Fig. 18. IF, through any two points (L, M) in the curve, right lines (LV, MT) be drawn parallel to the aſymptotes, then the rectangles under each of theſe lines and the adjacent diſtance (on the aſymptote) from the center, ſhall be equal; *i. e.* $LV \times VC = MT \times TC$.

DEMONSTRATION.

Through the points L, and M, draw the right line LM, and let $RL = y$, $LV = d$, $RV = p$, $MQ = z$, $QT = x$, $MT = c$, $VC = b$, and $TC = a$. Then, becauſe of parallels, the $\triangle$s RVL, RCQ, MTQ are ſimilar. But (by *Prop.* 49.) $y = z$, $\therefore c = p$, and $x = d$, and c ($= p$:) $d ::$ ($p + b =$) $c + b :$ ($a + x =$) $a + d$; $\therefore ca = db$, or $LV \times VC = MT \times TC$. *Q. E. D.*

COROLLARY I.

Hence if the lines MT, *rs*, *qt*, *&c.* be drawn parallel to the aſymptote CR, and the parallelograms T*m*, S*n*, *to*, &c. be inſcribed, they will be equal to each other. Becauſe, by the ſame reaſoning, as in this *Prop.* we may prove each of them equal to the parallelogram LC.

COROLLARY II.

Each of the inſcribed parallelograms T*m*, S*n*, *&c.* is equal to the ſquare of a right line (as BS) drawn from the vertex B, parallel to the aſymptote CR.

For (by this *Prop.*) each of them is equal to $BS\ (=GC) \times SC$; but (by the *Geneſis*) the $\angle BCG = \angle BCS$, and (from parallels) $\angle BCG = \angle SBC$, $\therefore \angle BCS = \angle SBC$, and (by the 6 *E.* 1.) $BS = SC$; and conſequently each of the parallelograms T*m*, *sn*, *&c.* is equal to $BS \times BS$, or BSq.

SCHOLIUM.

Right lines drawn from one aſymptote, and parallel to the other, and terminated by the curve (as *tq*, *sr*, *&c.*) are called ordinates; and the diſtance of thoſe lines from the center (as *t*C, *s*C) abſciſſas, and a right line drawn from the vertex, parallel to the aſymptote (as BS) the parameter of the exterior hyperbola; and if p be put for ſuch parameter, x for the abſciſſa, and y for the ordinate, then (by the laſt *Corol.*) $pp = yx$.

PROPOSITION LIII.

IF, on either of the aſymptotes (as CF) from the center right lines be ſet off in continual proportion (as CD, CE, CF;) and if, from the extremities of theſe lines, there be drawn lines parallel to the other aſymptote, and continued to the curve (as DG, EH, FI) they ſhall likewiſe be in continual proportion; that is, if CD, CE, CF be ∺, then DG, EH, FI will be ∺. Fig. 19.

DEMONSTRATION.

By *Prop.* 52. $GD \times DC = HE \times CE$, and $CF \times FI = CE \times EH$; and (by ſuppoſition) $CD \times CF = CE \times CE$, $\therefore DG : EH :: (CE : DC :: CF : CE ::)\ EH : FI$. *Q. E. D.*

PROPOSITION LIV.

IF, on either aſymptote, there be ſet off equal parts from the center, that is, if right lines be ſet off from the center in continual arithmetical proportion (as CM, CN, CO, CP, &c.) and from the extremities of theſe, there be drawn right lines parallel to the other aſymptote, and continued to the curve (as MR, NS, OT, PV) theſe ſhall be in continued harmonic proportion.

DEMONSTRATION.

$MR\ (=CX) \times CM = NS \times CN = OT \times CO = VP \times CP$; therefore

Fig. 19.

$$\left.\begin{array}{l} CM : CN :: NS : MR \\ CM : CO :: TO : MR \\ CM : CP :: VP : MR \end{array}\right\} \text{but } CM = \left\{\begin{array}{l} \frac{1}{2}CN \\ \frac{1}{3}CO \\ \frac{1}{4}CP \end{array}\right\} \therefore \left\{\begin{array}{l} NS \\ TO \\ VP \end{array}\right\} = \left\{\begin{array}{l} \frac{1}{2} \\ \frac{1}{3} \\ \frac{1}{4} \end{array}\right\} MR,$$

and if MR be equal to 1, then $NS=\frac{1}{2}$, $OT=\frac{1}{3}$, $VP=\frac{1}{4}$, which (being the reciprocals of continued arithmetical proportion) are in continued harmonic proportion. *Q. E. D.*

PROPOSITION LV.

Fig. 20. IF, from the center on either aſymptote, there be ſet three continual proportionals (as CD, CE, CF) and from their extremities right lines be drawn parallel to the other aſymptote, and continued to the curve (as DG, EH, FI,) and if on the curve thro' the ends of the extreams (as I, G,) a right line be drawn as LM, then, I ſay, a right line drawn from the center through (H) the end of the mean, ſhall biſect that line; that is, CO biſects IG in O.

DEMONSTRATION.

Draw HK parallel to LM; then (from ſimilar $\triangle$s) $EH : KH :: FI : LI$; and $EH : KH :: DG : GL$, $\therefore EHq : HKq :: FI \times DG : LI \times GL$. But (by *Prop.* 53.) $EHq = FI \times DG$, $\therefore HKq = LI \times GL$; and (by *Prop.* 46.) KH is a tangent to the point H, and conſequently $IO = OG$, being parallel to it, is an ordinate to the diameter CO. *Q. E. D.*

PROPOSITION LVI.

Fig. 20. IF CD, CE, CF on the aſymptote (and conſequently by *Prop.* 53. DG, EH, FI) be in continual proportion; then the ſpaces (HEDG, EHFI) between the curve and aſymptotes on each ſide of the mean (EH) to the extreams (FI and DG) ſhall be equal.

DEMONSTRATION.

1. Through the points I and G, draw the right line LM, and, through the center and H, draw the right line CO; then (by *Prop.* 55 and 49.) $LO = OM$, $\therefore$ (by 1 *E.* 6.) $\triangle MOC = \triangle OCL$. But the ſpace OGH = the ſpace OHI, becauſe each is compos'd

of an indefinite number of equal ordinates, conſequently the ſpace C H G M = ſpace C H I L; and from each take away the △s M G P + N H C = △s F L I + H C E, then there remains the ſpace N H G P = ſpace E H F I.

2. But ▯C G ∥ ▯E N by *Prop.* 52, ∴ ▯N G = ▯R E, and conſequently the ſpace H E D G = (ſpace H R G + ▯R E = ſpace H R G + ▯N G = ſpace N H G P = by the 1. part) ſpace E H F I. *Q. E. D.*

PROPOSITION LVII.

IF on either aſymptote be ſet off continual proportionals, and from their extremities right lines be drawn parallel to the other aſymptote, then the ſpaces between theſe lines ſhall be as the logarithms of the ratios of the lines which bound them: that is, if C*a*, C*b*, C*c*, C*d*, *&c.* be ∺, then the ſpace *ackf* is as the logarithm of the ratio of *ck* to *af*; and the ſpace *adgf*, as the logarithm of the ratio of *dg* to *af*, *&c.* Fig. 20.

DEMONSTRATION.

Let the ſpaces between the parallels be A, B, C, D, *&c.* (as in the figure;) then (by ſuppoſition) $\frac{\mathrm{C}a}{\mathrm{C}b} = \frac{\mathrm{C}b}{\mathrm{C}c}$, ∴ (by *Prop.* 56.) A = B, and $\frac{\mathrm{C}c}{\mathrm{C}d} = \frac{\mathrm{C}d}{\mathrm{C}e}$, ∴ B = C, *&c.* that is, if $\frac{\mathrm{C}a}{\mathrm{C}b} = \frac{\mathrm{C}b}{\mathrm{C}c} = \frac{\mathrm{C}c}{\mathrm{C}d} = \frac{\mathrm{C}d}{\mathrm{C}e}$, *&c.* then A = B = C = D, *&c.* whence the ſpaces are a ſeries of continued arithmetical proportionals, fitted to a ſeries of continued geometrical proportionals, and conſequently the addition of one anſwers to the multiplication of the other, which is the property of logarithms; as, for example,

Multiply the geometrical ſeries $\frac{\mathrm{C}a}{\mathrm{C}b} = \frac{\mathrm{C}b}{\mathrm{C}c}$, the product will be $\left(\frac{\mathrm{C}a}{\mathrm{C}c}\right.$ = by *Prop.* 52.) $\frac{ck}{af}$; and add the correſponding arithmetical ſeries, and the ſum is (A + B =) the ſpace *ackf*; conſequently the ſpace *ackf*, is as the logarithm of the ratio of *ck* to *af*. *Q. E. D.*

PROPOSITION LVIII.

THE areas of two hyperbolas, having the ſame tranſverſe axis, are as their conjugates.

DEMONSTRATION.

Let F B, *f*B, be two hyperbolas deſcribed to the ſame tranſverſe axis Fig. 21.

axis AB; then (by *Prop.* 1.) GFq : BGA :: BDq : BCq, and G*f*q : BGA :: B*d*q : BCq, ∴ (by equality) GFq : G*f*q :: BDq : B*d*q, and (by 22 *E.* 6.) GF : G*f* :: BD : B*d*. But the ſum of all the GF, G*f* do reſpectively conſtitute the areas of the hyperbolic ſpaces BFG, B*f*G; therefore (by 12 *E.* 5.) theſe areas are as the conjugate axes. *Q. E. D.*

PROPOSITION LIX.

PArallelograms circumſcribing any diameters of an hyperbola are equal.

DEMONSTRATION.

From the vertex of the diameter, and of the curve, draw FI, BH, parallel to the aſymptote CQ, and to the other aſymptote let fall the perpendiculars FG, BD. Put $IC=x$, $FI=y$, $BD=c$. and $CH=a$; then (from ſimilar △s) FP : FQ :: PI : IC; but PF=FQ, ∴ PI=IC, and $CP=2x$, and FG (from the ſimilar △s HBD, GIF, is) $=\frac{cy}{a}$; ∴ the area of the parallelogram PFCK $=PC\times FG=\frac{2cyx}{a}=$ (becauſe, by *Scholium* to *Prop.* 52. $yx=a^2$,) $2ac=CE\times BD=EBC$. *Q. E. D.*

APPENDIX.

Having conſidered the properties of the *Parabola*, *Ellipſe*, and *Hyperbola*, from their conſtruction *in plano*, without any regard had to the *Cone*, I ſhall ſubjoin the properties of the three figures made by the cutting a *Cone* by a plane; which properties, from their being the ſame with thoſe before delivered, plainly prove the figures, whoſe properties I have deſcribed, to be the true ſections of a *Cone*.

If a cone *be cut by any plane, to find the figure of the ſection.*

Fig. 24, 25, & 26. of Plate 2d. of the *Ellipſe.*

*LET ABC be a *cone* ſtanding on a circular baſe BC, and IEM its ſection ſought; and let KILM be any other ſection parallel to the baſe, and meeting the former ſection in HI; and ABC, a third ſection, perpendicularly biſecting the former in EH and KL, and the *cone* in the triangle ABC; and producing EH (in *Fig*. 25.) till it meet AK in D, and having drawn EF and DG parallel to KL, and meeting AB and AC in F and G, call EF$=a$, DG$=b$, ED$=c$, the abſciſſa EH$=x$, and the ordinate HI$=y$; and by reaſon of the ſimilar triangles EHL, EDG, ED : DG :: EH : HL$=\frac{bx}{c}$; then by reaſon of the ſimilar triangles DEF, DHK, DE : EF :: DH : ($c-x$ in *Fig*. 24. and $c+x$ in *Fig*. 25.) HK$=\frac{ac\mp ax}{c}$; laſtly, ſince the ſection KIL is parallel to the baſe, and conſequently circular, HK×HL will be =HIq, that is, $\frac{abcx\mp abx^2}{c^2}=y^2$; and if p be a fourth proportional to c, a and b, then $\frac{ab}{c}$ equal p, and (by ſubſtitution) $\frac{pcx\mp px^2}{c}=y^2$; and if X, and Y, be put for any other abſciſſa, and ordinate, then by the ſame reaſoning it may be proved that $\frac{pcX\mp X^2}{c}=Y^2$. Hence

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* See *Prop*. 18. of Sir *Iſaac Newton*'s Algebra.

A P P E N D I X.

1. When a *cone* is cut by a plane which interſects both its ſides (as in *Fig.* 24.) then the property of the curve, made by the plane of that ſection, will be ſuch, that $\overline{c-x}\times x : y^2 :: (c : p ::)\ \overline{c-X}\times X : Y^2$; which is the ſame property with *Corol.* to *Prop.* 2. of the *Ellipſe* foregoing.

2. When a plane cuts the baſe and ſide of a *cone* continued from the vertex (as in *Fig.* 25.) the property of the curve, made by the plane of that ſection, will be ſuch, that $\overline{c+x}\times x : y^2 :: (c : p ::)\ \overline{c+X}\times X : Y^2$; which is the ſame with the preceding *Corol.* to *Prop.* 2. of the *Hyperbola.*

If a *cone* be cut by a plane parallel to one of its ſides (as in *Fig.* 26.) and if AF be $=a$, HK$=b$, EH the abſciſſa $=x$, and the ordinate IH$=y$, then (by reaſon of the ſimilar triangles AFE, EHL) AF : (FE=) KH :: EH : HL$=\frac{bx}{a}$; but KH×HL=HIq, that is, $\frac{b^2x}{a}=y^2$; and if you make p, a third proportional to a, and b, then $\frac{b^2}{a}=p$, and (by ſubſtitution) $px=y^2$; and if you put X, and Y, for any other abſciſſa and ordinate, then by the ſame manner of reaſoning it may be proved that $pX=Y^2$; whence, $Y^2 : y^2 :: (pX : px ::)$ X : x, which is the ſame with the preceding *Cor.* to *Prop.* 1. of the *Parabola*,

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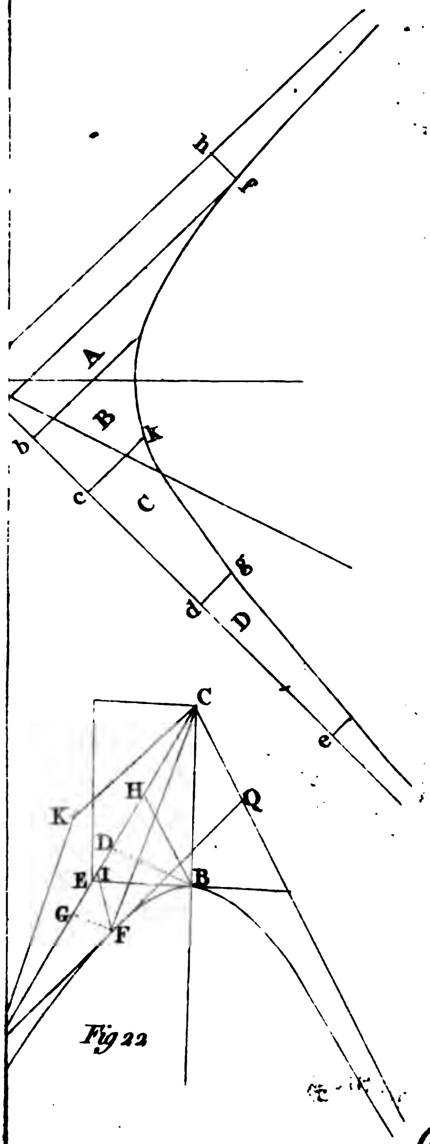
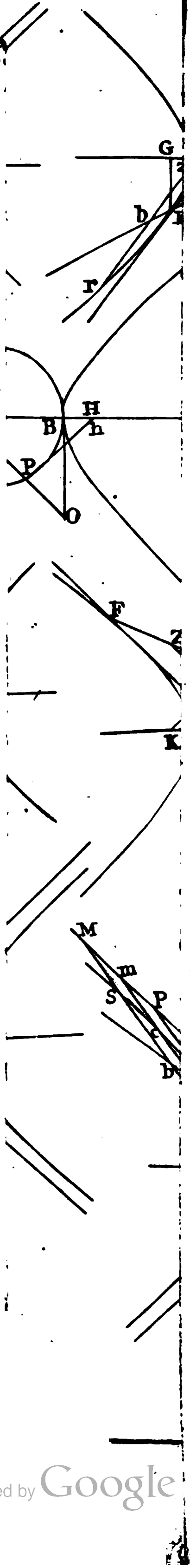


Fig 22



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